Measurement of Inequality and Social Welfare

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Ranking income distribution and Lorenz curves: Partial and complete orderings

(i) Partial orderings: Stochastic and inverse stochastic dominance, Lorenz dominance
(ii) Complete orderings:

a. Social welfare criteria based on expected utility theory
b. Rank-dependent social welfare criteria

Important issue in both policy work, descriptive analysis and causal inference:

1. Statistical offices and gov agencies compare distribution functions and Lorenz curves across countries, subgroups and time
2. Research compares distributions of earnings, income, consumption and wealth to evaluate economic policies and social welfare

The cumulative distribution function and its inverse

Let F be a member of the set $\mathscr{F}$ of cumulative distribution functions with mean μ_F and left inverse defined by

$$F^{-1}(t) = \inf\{x : F(x) \geq t\}$$

Note that both discrete and continuous distribution functions are allowed in $\mathscr{F}$, and though the former is what we actually observe, the latter often allows simpler derivation of theoretical results and is a valid large sample approximation. Thus, in most cases below F will be assumed to be a continuous distribution function, but the assumption of a discrete distribution function will be used where appropriate. To fix ideas, we will refer to F as the income distribution, although our framework can be applied to any type of distribution functions.

In order to rank distribution function we introduce the ordering relation $\succeq$

Ranking distribution functions: Examples

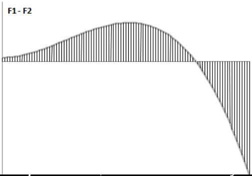
Suppose we want to rank two distributions, F_1 and F_0

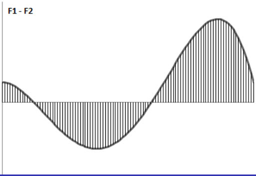
- Assume that the ordering relation $\succeq$satisfies first-degree stochastic dominance, i.e.

$$F_1(x) \leq F_0(x) \;\; \textit{for all } x \in [0,\infty) \Leftrightarrow F_1^{-1}(t) \geq F_0^{-1}(t) \;\; \textit{for all } t \in [0,1].$$

Can be used as a ranking criterion when distribtion don't cross. But how do we deal with intersecting distribution functions (Figures 2 and 3)?

Conventional approach in empirical work: Using summary measures like the mean, the median and the variance or weighted means.





Second-degree stochastic and inverse stochastic dominance

Definition

A distribution function F_1 is said to *second*-degree *stochastic dominate* a distribution function F_0 if and only if

$$\int_0^y F_1(x)dx \leq \int_0^y F_0(x)dx \text{ for all } y \in [0, \infty)$$

and the inequality holds strictly for some $y \in (0, \infty)$.
A distribution function F_1 is said to *second*-degree *inverse stochastic dominate* a distribution function F_0 if and only if

$$\int_0^u F_1^{-1}(t)dt \geq \int_0^u F_0^{-1}(t)dt \text{ for all } u \in [0, 1]$$

and the inequality holds strictly for some $u \in (0, 1)$.

As was demonstrated by Atkinson (1970), second-degree stochastic dominance is equivalent to second-degree inverse stochastic dominance, which is called generalized Lorenz dominance by Shorrocks (1983).

Moreover, *under the restriction of equal mean incomes second degree inverse stochastic dominance is equivalent to the criterion of non-intersecting Lorenz curves*.

Third-degree stochastic dominance

Since situations where second-degree dominance does not provide unambiguous ranking of distribution functions may arise, it will be useful to introduce weaker ranking criteria than second-degree dominance. To this end it appears attractive to consider third-degree stochastic and inverse stochastic dominance.

Definition

A distribution function F_1 is said to *third*-degree *stochastic dominate* a distribution function F_0 if and only if

$$\int_0^z \int_0^y F_1(x)dxdy \leq \int_0^z \int_0^y F_0(x)dxdy \text{ for all } z \in [0,\infty) \Leftrightarrow$$

$$\int_0^z (z-x)(F_1(x) - F_0(x))\,dx \leq 0 \text{ for all } z \in [0,\infty)$$

Third-degree inverse stochastic dominance

Definition

A distribution function F_1 is said to *third*-degree *inverse stochastic dominate* a distribution function F_0 if and only if

$$\int_0^v \int_0^u F_1^{-1}(t)dtdu \geq \int_0^v \int_0^u F_0^{-1}(t)dtdu \text{ for all } v \in [0,1] \Leftrightarrow$$

$$\int_0^v (v-t)\left(F_1^{-1}(t) - F_0^{-1}(t)\right) dt \leq 0, \text{ for all } v \in [0,1]$$

and the inequality holds strictly for some $v \in (0,1)$.

Note that third-degree stochastic and inverse stochastic dominance ***do not*** coincide.

Transfer principles associated with second- and third-degree dominance

Definition

(The Pigou-Dalton principle of transfers). Consider a discrete income distribution F . A transfer $\delta > 0$ from a person with income $x + h$ (*or* $F^{-1}(t)$) to a person with income x (*or* $F^{-1}(s)$) is said to reduce inequality in F when $h > 0$ ($s < t$) and to raise inequality in F when $h < 0$ ($s > t$).

(i) If $\mu_{F_1} = \mu_{F_0}$, the condition of *second-degree inverse stochastic dominance* is identical to *the Pigou-Dalton transfer principle.*

Definition

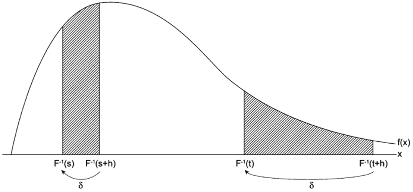
(The principle of diminishing transfers, Kolm,1976). Consider a discrete income distribution F . A transfer $\delta > 0$ from a person with income $x + h_1$ to a person with income x is said to reduce inequality in F more than a transfer δ from a person with income $x + h_1 + h_2$ to a person with income $x + h_2$.

(ii) If $\mu_{F_1} = \mu_{F_0}$, the condition of *third-degree inverse stochastic dominance* is identical to *the principle of diminishing transfers.*

Rank-preserving transfers

Definition

(The principle of positional transfer sensitivity, Mehran, 1976). Consider a discrete income distribution F . A rank-preserving transfer $\delta > 0$ from a person with income $F^{-1}(s+h)$ to a person with income $F^{-1}(s)$ is said to have a stronger equalizing effect on F than a transfer $\delta > 0$ from a person with income $F^{-1}(t+h)$ to a person with income $F^{-1}(t)$ when $s < t$.



(iii) If $\mu_{F_1} = \mu_{F_0}$, the condition of *third-degree inverse stochastic dominance* is identical to *the principle of first-degree downside positional transfer sensitivity*.

Lorenz dominance

Definition

A Lorenz curve L_1 is said to first-degree dominate a Lorenz curve L_0 if
$L_1(u) \geq L_0(u)$ *for all* $u \in [0,1]$

and the inequality holds strictly for some $u \in [0,1]$.

First-degree Lorenz dominance is identical to *the Pigou-Dalton transfer principle*. A social planner who prefers the dominating one of non-intersecting Lorenz curves favors transfers of incomes which reduce the differences between the income shares of the donor and the recipient, and is therefore said to be *inequality averse*.

Second-degree Lorenz dominance

To deal with situations where Lorenz curves intersect a weaker principle than first-degree Lorenz dominance is called for. To this end it is normal to employ second-degree upward Lorenz dominance defined by

Definition

A Lorenz curve L_1 is said to second-degree upward dominate a Lorenz curve L_0 if
$\int_0^u L_1(t)dt \geq \int_0^u L_0(t)dt$ *for all* $u \in [0,1]$

and the inequality holds strictly for some $u \in [0,1]$.

Second-degree upward Lorenz dominance is identical to *the principle of first-degree downside positional transfer sensitivity*. *Under the restriction of equal mean incomes third-degree (upward) inverse stochastic dominance is equivalent to the criterion of seond-degree upward Lorenz dominance*.

Higher degrees of inverse stochastic dominance and Lorenz dominance

Since situations where second-degree (upward or downward) inverse stochastic and Lorenz dominance do not provide unambiguous ranking of distribution functions and Lorenz curves may arise, it is useful to introduce weaker dominance criteria than third-degree inverse stochastic dominance and second-degree Lorenz dominance. To this end two hierarchical sequences of nested inverse stochastic (Lorenz dominance) criteria might be introduced; one departs from third-degree upward inverse stochastic dominance (second-degree upward Lorenz dominance) and the other from third-degree downward inverse stochastic dominance (downward Lorenz dominance). More on this in Aaberge (2009, SCW).

Complete orderings: Social welfare criteria based on expected utility theory

The problem of ranking income distributions formally corresponds to the problem of choosing between uncertain prospects. This relationship has been utilized by e.g. Kolm (1969) and Atkinson (1970) to characterize the criterion of second order (upward) stochastic dominance. Atkinson reinterpreted the standard theory of choice under uncertainty and demonstrated that inequality aversion can in fact be viewed as being equivalent to risk aversion. This was motivated by the fact that in cases of equal mean incomes the criterion of non-intersecting Lorenz curves is equivalent to second-degree stochastic dominance, which means that the Pigou-Dalton transfer principle is identical to the principle of mean preserving spread introduced by Rothschild and Stiglitz (1970). To choose between F_0 and F_1 we can then use the following criterion

$$\int_0^\infty u(x)dF_1(x) \geq \int_0^\infty u(x)dF_0(x)$$

Axiomatic justification of the primal approach

Assume that the preference relation $\succeq$ of the social planner satisfies the following axioms:

(Order). $\succeq$ is a transitive and complete ordering on $\mathscr{F}$.

(Continuity). For each $F \in \mathscr{F}$ the sets $\{F^{\star} \in \mathscr{F} : F \succeq F^{\star}\}$and $\{F^{\star} \in \mathscr{F} : F^{\star} \succeq F\}$ are closed (w.r.t. L_1-norm).

(Dominance). Let $F_0, F_1 \in \mathscr{F}$. If $F_1^{-1}(t) \geq F_0^{-1}(y)$ for all $t \in [0,1]$ and the inequality holds strictly for some t$\in (0,1)$ then $F_1 \succeq F_0$.

(Independence). Let F_0, F_1 and F_2 be members of $\mathscr{F}$ and let $\alpha \in [0,1]$. Then $F_1 \succeq F_0$ implies $(\alpha F_1(x)+(1-\alpha)F_2(x)) \succeq (\alpha F_0(x)+(1-\alpha)F_2(x))$.

Von Neuman and Morgenstern (1936) proved that a preference relation $\succeq$ that satisfies Axioms 1-4 can be represented by the following family of social welfare functions

$$Eu(X) = \int u(x)dF(x)$$

Measures of inequality based on the primal approach

Atkinson (1970) proposed to use

$$I_u(F) = 1 - \frac{u^{-1}(Eu(X))}{\mu}$$

as a measure of inequality, where $u^{-1}(Eu(X))$ is denoted *the equally distributed equivalent income*

and

$\mu I_u(F)$ is measure of the loss in social welfare due to inequality in the distribution F.

The relationship between dominance criteria and primal social welfare functions

Theorem

Let F_1 and F_0 be members of F. Then the following statements are equivalent.
(i) F_1 second-degree upward inverse stochastic dominates F_0
(ii) $E_{F_1} u(X) > E_{F_0} u(X)$ for all increasing concave u

Theorem

Let L_1 and L_0 be members of L. Then the following statements are equivalent.
(i) L_1 first-degree dominates L_0
(ii) $I_u(F_1) < I_u(F_0)$ for all increasing concave u

Complete orderings: The family of rank-dependent social welfare functions

The general family of rank-dependent measures of social welfare introduced by Yaari (1987,1988) is defined by

$$W_P(F) = \int_0^1 P'(t)F^{-1}(t)dt,$$

and can be interpreted as the equally distributed equivalent income. The weighting function P' is the derivative of a preference function that is a member of the following the set of preference functions:

$$\begin{aligned} \mathscr{P} &= \{P : P'(t) > 0 \textit{ and } P''(t) < 0 \\ &\quad \textit{for all } t \in (0,1),\, P(0) = P'(1) = 0,\, P(1) = 1\} \end{aligned}$$

- W_P preserves 1st-degree dom, since $P'(t) > 0$, and
  - W_P preserves 2nd-degree dom (and Pigou-Dalton), since $P''(t) < 0$
    - $W_P \leq \mu_F$, and $W_P = \mu_F$ iff F is the egalitarian distribution

Complete orderings: The family of rank-dependent social welfare functions

The general family of rank-dependent measures of social welfare introduced by Yaari (1987,1988) is defined by

$$W_P(F) = \int_0^1 P'(t)F^{-1}(t)dt,$$

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- W_P preserves 1st-degree dom, since $P'(t) > 0$, and
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    - $W_P \leq \mu_F$, and $W_P = \mu_F$ iff F is the egalitarian distribution

Dual measures of inequality

Since $W_P(F)$ can be interpreted as the equally distributed equivalent income the dual family of inequality measures is defined by

$$J_P(F) = 1 - \frac{W_P(F)}{\mu},$$

where $\mu = EX = \int x dF(X)$.
Note that $\mu J_P(F)$ is a measure of the loss in social welfare due to inequality in the distribution F.

Axiomatic justification of the dual approach

Assume that the preference relation $\succeq$ of the social planner satisfies the following axioms:

(Order). $\succeq$ is a transitive and complete ordering on $\mathscr{F}$.

(Continuity). For each $F \in \mathscr{F}$ the sets $\{F^\star \in \mathscr{F} : F \succeq F^\star\}$and $\{F^\star \in \mathscr{F} : F^\star \succeq F\}$ are closed (w.r.t. L_1-norm).

(Dominance). Let $F_0, F_1 \in \mathscr{F}$. If $F_1^{-1}(t) \geq F_0^{-1}(y)$ for all $t \in [0,1]$ and the inequality holds strictly for some t$\in (0,1)$ then $F_1 \succeq F_0$.

(Independence). Let F_0, F_1 and F_2 be members of $\mathscr{F}$ and let $\alpha \in [0,1]$. Then $F_1 \succeq F_0$ implies
$\left(\alpha F_1^{-1}(t) + (1-\alpha)F_2^{-1}(t)\right)^{-1} \succeq \left(\alpha F_0^{-1}(t) + (1-\alpha)F_2^{-1}(t)\right)^{-1}$.

Yaari (1936) proved that a preference relation $\succeq$ that satisfies Axioms 1-4 can be represented by the following family of social welfare functions

$$W_P(F) = \int_0^1 P'(t)F^{-1}(t)dt,$$

Normative justification of the general family

The normative justification of W_P can be made in terms of a

(a) Theory for ranking distribution functions:

- With basic ordering and continuity assumptions, the dual independence axiom characterizes W_P (Yaari, 1988)

(b) Value judgement of the trade-off between the mean and (in)equality in the distributions (Ebert, 1987; Aaberge, 2001)

$$W_P = \mu_F[1 - J_P(F)]$$

where μ_F is the mean of F and
$J_P(F)$ is the family of rank-dependent measures of inequality aggregating the P'-weighted Lorenz curve of F

The Gini subfamily

If we choose

$$P_{1k}(t) = 1 - (1-t)^{k-1}, \, k > 2$$

then W_P is equal to the extended Gini family of social welfare functions (Donaldson and Weymark, 1980)

$$W_{G_k} \quad = \quad \mu \left[1 - G_k(F) \right] =, \qquad k > 2$$

where

- $G_k(F)$ is the extended Gini family of inequality measures
- $G_3(F)$ is the Gini coefficient and $W_{G_2} = \mu$
- Note that $\{\mu, W_{G_i}(F) : i = 3, 4, \ldots\}$ uniquely determines the distribution function F (Aaberge, 2000)

The Lorenz subfamily

If we instead choose

$$P_{2k}(t) = \frac{(k-1)\,t - t^{k-1}}{k-2}, \; k > 2$$

then W_P is the Lorenz family of social welfare functions (Aaberge, 2000)

$$W_{D_k} \quad = \quad \mu\left[1 - D_k(F)\right], \quad k > 2$$

where

- $D_k(F)$ is the Lorenz family of inequality measures
- $D_3(F)$ is the Gini coefficient
- Note that $\{\mu, W_{D_i}(F) : i = 3, 4, ...\}$ uniquely determines the distribution function F (Aaberge, 2000)

The relationship between dominance criteria and dual criteria of social welfare and inequality

Theorem

Let F_1 and F_0 be members of F. Then the following statements are equivalent.

(i) F_1 second-degree upward inverse stochastic dominates F_0

(ii) $\int_0^1 P'(t)F_1^{-1}(t)dt > \int_0^1 P'(t)F_0^{-1}(t)dt$ for all increasing concave P $(P''(t) < 0)$

Theorem

Let L_1 and L_0 be members of L. Then the following statements are equivalent.

(i) L_1 first-degree dominates L_0

(ii) $J_P(F_1) < J(F_0)$ for all increasing concave P

Partial dual ordering - Third degree upward dominance

Note that second degree inverse stochastic dominance is defined by

$$\Lambda_F^2(u) \equiv \int_0^u F^{-1}(t)dt, \quad u \in [0,1]$$

To define third degree upward inverse stochastic dominance, we use the notation

$$\Lambda_F^3(u) \equiv \int_0^u \Lambda_F^2(t)dt = \int_0^u (u-t)F^{-1}(t)dt, \quad u \in [0,1]$$

Definition

A distribution F_1 is said to *third* degree *upward inverse stochastic dominate* a distribution F_0 if and only if

$$\Lambda_{F_1}^3(u) \geq \Lambda_{F_0}^3(u) \text{ for all } u \in [0,1]$$

and the inequality holds strictly for some $u \in (0,1)$.

Transfer principle

$\Delta_s W_P(\delta, h)$: change in W_P of a fixed progressive transfer δ from an individual with rank $s+h$ to an individual with rank s.
$\Delta^1_{st} W_P(\delta, h) \equiv \Delta_s W_P(\delta, h) - \Delta_t W_P(\delta, h)$.

Definition

(Zoli, 1999; Aaberge, 2000, 2009) W_P satisfies the principle of first degree downside positional transfer sensitivity (DPTS) if and only if

$$\Delta^1_{st} W_P(\delta, h) > 0, \quad \text{when } s < t.$$

Equivalence result

Let $\mathscr{P}_3$ be the family of preference functions defined by

$$\mathscr{P}_3 = \left\{ P \in \mathscr{P} : P'''(t) > 0, \right\}$$

Theorem

Let F_1 and F_0 be members of F. Then the following statements are equivalent.
(i) F_1 third-degree upward inverse stochastic dominates F_0
(ii) $W_P(F_1) > W_P(F_0)$ for all $P \in \mathscr{P}_3$
(iii) $W_P(F_1) > W_P(F_0)$ for all $P \in \mathscr{P}$ where W_P satisfies first-degree DPTS

- $\Rightarrow$ (i) and (ii): least-restrictive set of social welfare functions that unambiguously rank in accordance with 3-UID
- $\Rightarrow$ (i) and (iii): normative justification for 3-UID