Distributive Justice and Economic Inequality

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UiO, January 17th, 2017

Framework

A **society** consists of a finite set of individuals $N \equiv \{1, ..., n\}$.

The set of possible social decisions, or **alternatives**, is X and consists of at least 3 alternatives.

A **ranking** over X is denoted R. (As usual the strict counterpart of R is denoted P; indifference is I)

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Each individual has a preference $R_i \in \mathscr{R}$.

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Bergson (1938) writes that the social ranking R can be represented by a function:

$$W = W(\{U_i\}),$$

where U_i is a representation of each preference R_i (*a utility function?*).

He clarifies that if a different U_i is used, W should change accordingly so that the social ranking is unchanged.

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Bergson

The approach, it must also be noted, requires a group of value propositions additional to those I have presented. Insofar as the Cambridge economists require that the economic welfare of the community be an aggregate of individual welfares, value judgments must be introduced to the effect that each individual contributes independently to the total welfare. These value propositions, which imply the complete measurability of the economic welfare function aside from an arbitrary origin and a scalar constant, are not necessary for the derivation of the maximum conditions, and accordingly are not essential to the analysis. [Bergson, 1938, QJE]

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The framework is unchanged. But the goal is now different.

As already said, a **preference profile** is a tuple of individual preferences $\{R_i\} \in \mathscr{R}^n$, each defined over X.

Let S be the **domain** of possible societies: it defines all admissible combinations of N, X, and $\{R_i\}$.

An **Arrowian SWF** is a function f (or a collective choice rule) that associates a social ranking to each society in S.

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Gibbard: Socrates meets Meletus

SOCRATES. Good morning, Meletus. That is a beautiful ice cream cone you have. Are you planning to eat it?

MELETUS. Indeed, Socrates.

SOCRATES. Yet are you not a man of justice, Meletus?

MELETUS. The most just man in ancient Greece, Socrates!

SOCRATES. Then if I can show you that justice demands that you let me eat it, you will certainly let me.

MELETUS. If you could show such a thing, I would give you the cone. But since it is mine and I want to eat it myself, I don't see how you can possibly show that justice demands that I let you eat it.

Gibbard: Socrates meets Meletus

SOCRATES. I admit the task will not be easy. But tell me, Meletus, do you accept that justice is determined by a social choice function which satisfies the conditions of collective rationality, unanimity, and independence of irrelevant alternatives?

MELETUS. Of course, Socrates.

SOCRATES. Now Meletus, suppose for some reason you could not eat the ice cream cone, and I wanted to eat it. Do you agree that the social choice should be to let me eat it, regardless of your preference in the matter?

MELETUS. Well I don't know, Socrates. It's my cone, after all.

Gibbard: Socrates meets Meletus

SOCRATES. Certainly it is your cone, and possibly we shall decide that that gives you the right to eat it if you can and want to. But if you can't eat it, our choice function shouldn't allow you to throw it away out of pure spite when I want it.

MELETUS. All right, I accept what you say, Socrates.

SOCRATES. Then if I prefer eating it to throwing it away, the social preference will rank my eating it above throwing it away, regardless of your preferences.

MELETUS. That seems a small thing to concede, especially since in fact I can eat it, and I want to, and there is no reason to suppose that a just social choice function would deny me my first choice in the matter.

Gibbard: Socrates meets Meletus

SOCRATES. We shall see. But now, Meletus, suppose I preferred to eat the cone, but preferred throwing it away to letting you eat it. And suppose you were not hungry and preferred to throw the cone away, but would rather eat it than let me eat it.

MELETUS. My preferences would never be so malicious, and I am surprised that you admit that yours might be!

SOCRATES. I was not accusing you or myself of malice. But malicious preferences are not impossible, and our social choice function must be able to handle them. Suppose, then, we had the malicious preferences I have described...

Gibbard: Socrates meets Meletus

...Then I would prefer eating the cone to throwing it away, and we have agreed that my preferences rule between those two alternatives, whatever your preferences may be. Furthermore, given the preferences I have pictured, we would both prefer throwing the cone away to having you eat it. Thus the social choice function would rank throwing it away above your eating it in the case I have pictured. Thus given these supposed individual preferences, justice would rank my eating the cone above throwing it away and throwing it away above your eating it. By transitivity, it would rank my eating it above your eating it.

Gibbard: Socrates meets Meletus

MELETUS. Your logic is impeccable, Socrates. But fortunately my preferences are not those in the supposition.

SOCRATES I have not forgotten that. But see how things would be if our preferences were as I have pictured them. I would prefer eating the cone to letting you eat it, you would prefer eating it to letting me eat it, and justice would demand that I eat it.

MELETUS. Yes, Socrates.

SOCRATES. But the situation as it stands now is exactly as I have just described...

Gibbard: Socrates meets Meletus

... I prefer eating it to letting you eat it and you prefer eating it to letting me eat it. In deciding whether you or I shall eat it, how we feel about throwing it away is irrelevant. If justice demanded that I eat the cone given the preferences I pictured earlier, it demands that I eat the cone given our actual preferences, for our actual preferences differ from those hypothetical preferences only in the way we rank throwing the cone away, which is not in question here.

MELETUS. The ice cream has melted, Socrates.

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What conditions to impose on f?

- Arrow proposes the following:
  - Completeness of R.
  - Transitivity of R.
  - Unrestricted domain.
  - Weak Pareto principle (*unanimity*).
  - Nondictatorship.
  - Independence of irrelevant alternatives.

Collective rationality

- The first two are well-known:
  - Completeness of R.
  - Transitivity of R.

Unrestricted domain (U)

- We defined S as the **domain** of possible societies: changing N, X, and $\{R_i\}$.
- Unrestricted domain requires that S includes all possible (finite) number of individuals, all possible number of alternatives (but at least 3 distinct!), all possible preferences.

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Weak Pareto (P).

- This axiom tells that if all individuals prefer alternative x to alternative y, so does society.

Formally, for each pair $x, y \in X$,

$$x \, P_i \, y \text{ for each } i \in N,$$

implies that

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Nondictatorship (D)

- There exist no individual that is so powerful that society always follows her/his preferences.

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Independence of irrelevant alternatives (I)

- The ranking of any pair of alternatives should only depend on how individuals rank these alternatives.

Formally, for each two societies S, S' and each pair $x, y \in X, X'$,

if individual preferences over x and y are unchanged,

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Arrow's result

Possibility theorem.

A SWF f can satisfy only 5 out of the following 6 axioms: *completeness*, *transitivity*, *unrestricted domain*, *weak Pareto*, *nondictatorship*, and *independence of irrelevant alternatives*.

Impossibility corollary.

There exists no SWF f satisfying all 6 axioms.

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Impossibility corollary.

There exists no SWF f satisfying all 6 axioms.

Sketch of the proof (see Sen, 2011)

- A set G of individuals is **decisive over the ordered pair** $\{x, y\}$, denoted $D_G(x, y)$, if and only if whenever all individuals in G prefer x to y, then society does as well (no matter what others prefer).
- If G is decisive on every ordered pairs, the G is **decisive**, denoted D_G.
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The field expansion lemma (see Sen, 2011)

- **If $D_G(x, y)$, then D_G.**
- *Proof.* Assume $D_G(x, y)$. We need to show that for each $a, b \in X$ such that all individuals in G prefer a to b, it follows that $D_G(a, b)$.
- There are several cases, depending on how a and b are ranked wrt x and y. But all are very similar. So assume that for each $j \in G$, aP_jx, xP_jy, and yP_jb.
- Assume also that for all other persons $i \notin G$, aP_ix and yP_ib. *[Why can we assume this?]*
- By weak Pareto, aPx and yPb. Since $D_G(x, y)$, xPy.
- By transitivity of R, aPb (no matter how those not in G rank them) and, thus, $D_G(a, b)$.
- Thus (after repeating for all cases), D_G.

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- **Assume D_G and G consists of more than 1 individual. Partition G into G_1 and G_2. Then, either D_{G1} or D_{G2}.**
- *Proof.* Assume for all $i \in G_1$, xP_iy and xP_iz and, for all $j \in G_2$, xP_jy and zP_jy. By D_G, xPy.
- If xPz, then G_1 is decisive over x, z and by the previous lemma D_{G1}. So assume not.
- Then, it must be that for some society, zRx. Transitivity and xPy, imply that zPy.
- Thus, G_2 is decisive over z, y and by the previous lemma D_{G2}.
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Concluding the proof (see Sen, 2011)

- By weak Pareto, if all individuals prefer x to y, so does society. Thus, $D_N(x,y)$ holds and, since this is true for each pair x, y, also D_N.
- If N consists of more than 1 individual, use Lemma 2.
- If the obtained decisive group consists of more than 1 individual, use Lemma 2 again.
- Since N is finite, there will be a decisive individual: a dictator.

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