Distributive Justice and Economic Inequality

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UiO, January 17th, 2017

Framework

- A pair $(x, i) \in X \times N$ is called a **station**.
- An evaluation **profile** is denoted U. It collects all individual evaluations on $X \times N$.
- If X is finite, U can be described by a matrix $|X| \times |N|$ with generic element $U(x, i)$.
- $U_x \equiv U(x, \cdot)$ is the row vector and $U_i \equiv U(\cdot, i)$ is the column vector.
- Let $\mathscr{U} \equiv \{U | U : X \times N \to \mathbb{R}\}$. A **domain** is a subset $\mathscr{D} \subseteq \mathscr{U}$.
- A social welfare functional is a map $F : \mathscr{D} \to \mathbb{R}$. It assigns social preferences R_U to each $D \in \mathscr{D}$, i.e. $R_U = F(U)$.

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Discussion

- We already discussed the relationship between BS-SWF and Arrowian SWF:
  - the Arrowian SWF defines a BS-SWF for each possible society.
- The SWFL approach is even more general:
  - the SWFL framework includes information about a specific representation of preferences, i.e. U;
  - if this information is disregarded and only preferences are taken into account, then the SWFL is "equivalent" to an Arrowian SWF;
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How much information?

- How to measure utility information?
- There are two dimensions: **intrapersonal** and **interpersonal**.
- *Intrapersonal* comparisons of utilities ask:
  - what can we say if $U(x,i)=4$, $U(y,i)=2$?
  - what do we learn from $U(\bar{x},i)=5$ and $U(\bar{y},i)=3$?

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Ordinal, cardinal, and ...

- Assume the only thing we learn is that i prefers x to y (or $\bar{x}$ to $\bar{y}$). Then the only thing that matters are the ordinal preferences R_i of i.

- In other words, any strictly increasing transformation ϕ of U_i, i.e. $\bar{U}_i = \phi_i(U_i)$, gives us the same information. This utility is **ordinal.**

- Assume we also learn that the change in utility from x to $\bar{x}$ is as large as the change in utility from y to $\bar{y}$.

- Then, the specific U_i matters. In fact, this information is preserved under a smaller set of transformations ϕ_i: it needs to be positive affine ($\phi_i(U_i) = a_i + b_i U_i$ with $b_i > 0$). Then, this utility is **cardinal.**

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Non-comparable, fully comparable, and unit comparable

- *Interpersonal* comparisons of utilities ask:
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  - what do we learn from $U(\bar{x},i)=5$ and $U(\bar{x},j)=3$?
- The first case is that we do not learn whether i is better-off than j. Any transformations ϕ_i, ϕ_j preserve this property. Then, the utilities of i and j are **non-comparable**.
- The opposite case is that we learn exactly that i is better-off than j both at x and at $\bar{x}$. This is preserved only if $\phi_i = \phi_j$. Then, the utilities of i and j are **fully comparable**.
- An intermediate case, is that we do not learn whether i is better-off than j, but we learn that moving from x to $\bar{x}$ both individuals enjoy the same utility gain. This is preserved only if $\phi_i(\Delta U_i) = \phi_j(\Delta U_j)$. Then, the utilities of i and j are **unit comparable** (or comparable in terms of gains and losses).

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Combining inter- and intra-personal comparisons (I)

- **Ordinality and non-comparability.** Invariance to individual positive transformations $V_i = \varphi_i(U_i)$.
  - F satisfies *ordinality and non-comparability* if for each pair $U, V \in \mathscr{U}$ such that for each $i \in N$ $V_i = \varphi_i \circ U_i$, then $F(U) \equiv R_U = R_V \equiv F(V)$.
- **Co-ordinality (common ordinal scale).** Invariance to common increasing transformations $V_i = \varphi(U_i)$.
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A graphical representation

Formal welfarism

- Let $\mathscr{H}(X,\mathscr{D})$ be the evaluation space:
  $\mathscr{H}(X,\mathscr{D}) \equiv \left\{ r \in \mathbb{R}^{|N|} \,|\, \exists x \in X, \exists U \in \mathscr{D} \text{ such that } U_x = r \right\}$.
- A **social welfare ordering (SWO)** R^* is a ranking of profiles of utility levels.
- A social welfare functional F satisfies **formal welfarism** if there exists a SWO R^* such that:

$$\forall u,v \in \mathscr{H}(X,\mathscr{D}), \forall x,y \in X, \forall U \in \mathscr{D},$$

$$\langle u = U_x \text{ and } v = U_y \rangle \Rightarrow \langle uR^*v \text{ iff } xR_U y \rangle .$$

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Formal welfarism: a characterization

- **Pareto indifference**:

$$\forall U \in \mathscr{D}, \forall x, y \in X, x I_U y \text{ if } U_x = U_y.$$

- **Binary independence:**

$$\forall V \in \mathscr{D}, \forall x, y \in X, x R_V y \text{ if } \exists U \in \mathscr{D} \text{ such that } V_x = U_x, V_y = U_y \text{ and }$$

- **Theorem:** When $\mathscr{D} = \mathscr{U}$, formal welfarism is equivalent to the combination of *Pareto indifference* and *binary independence*. Moreover, $\mathscr{H}(X, \mathscr{D}) = \mathbb{R}^N$.

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- The "associated" utilitarian SWFL F requires that for each pair $x, y \in X$ and each $U \in \mathscr{D}$, $xR_U y$ if and only if

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Axioms

- **Weak Pareto***. For each pair $u, v \in \mathbb{R}^N$, if $u \gg v$ then uP^*v.
- First, define a **permutation** π and let Π be the set of all permutations.
- **Anonoymity***. For each $\pi \in \Pi$ and each pair $u, v \in \mathbb{R}^N$, uI^*v if $v = \pi u$.
- **Inv***$(a_i + bu_i)$. For each $(a_i) \in \mathbb{R}^N$, for each $b \in \mathbb{R}_+$, for each pair $u, v \in \mathbb{R}^N$,

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Theorem: utilitarianism

- **Theorem 4.4** (d'Aspremont and Gevers, 2002). A SWO $R^*$is pure *utilitarian* iff it satisfies *weak Pareto**, *Anonoymity**, and *Inv**$(a_i + bu_i)$.
- *Proof.* Two parts.
  - Necessity. A utilitarian SWO R^* satisfies *weak Pareto**, *Anonoymity**, and *Inv**$(a_i + bu_i)$.
  - Sufficiency. A SWO R^* that satisfies *weak Pareto**, *Anonoymity**, and *Inv**$(a_i + bu_i)$ is utilitarian.

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    - Sufficiency. A SWO R^* that satisfies *weak Pareto**, *Anonoymity**, and *Inv**$(a_i + bu_i)$ is utilitarian.

Proof: sufficiency

- Let a pair $u, v \in \mathbb{R}^{|N|}$ be such that $\sum_{i \in N} u_i = \sum_{i \in N} v_i$.
- By *Anonymity**, permute u and v in increasing order. Clearly, $\pi u I^* u$ and $\bar{\pi} v I^* v$.
- Subtract from each raw of πu and $\bar{\pi} v$ the smallest number. By *Inv**$(a_i + bu_i)$, the new utility vectors are rank as the starting ones.
- Repeat permutation and subtraction at most $|N|$ times, you get two vectors of zeros, which are equally good.

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- Assume now that that $\sum_{i \in N} u_i > \sum_{i \in N} v_i$.
- Define $\delta \equiv \frac{\sum_{i \in N} u_i - \sum_{i \in N} v_i}{|N|}$ and the utility vector w such that for each $i \in N$, $w_i = u_i - \delta$.
- Since $u \gg w$, *weak Pareto** implies that uP^*w.
- Since $\sum_{i \in N} w_i = \sum_{i \in N} v_i$, wI^*v. By transitivity, uP^*v.

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- The leximin SWO R^* holds that for each pair $u, v \in \mathbb{R}^N$, uR^*v if and only if

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Axioms

- **Strict Pareto*.** For each pair $u, v \in \mathbb{R}^N$, if $u > v$ then uP^*v.
- **Minimal Individual Symmetry*.** For any pair $i, j \in N$, there exists uI^*v such that $u_i > v_i$, $u_j < v_j$ and $u_k = v_k$ for all $k \neq i, j$.
- **Minimal equity*.** For some pair $i, j \in N$, there exists $u, v \in \mathbb{R}^N$, such that $u_k = v_k$ for each $k \neq i, j$, $v_i < u_i < u_j < v_j$, and uR^*v.
- **Inv*$(\phi(u_i))$.** For each real-valued and increasing function ϕ, for each pair $u, v \in \mathbb{R}^N$,

$$uR^*v \Leftrightarrow (\phi(u_1), ..., \phi(u_n)) R^* (\phi(v_1), ..., \phi(v_n)).$$

- **Separability*.** For each $u, v, u', v' \in \mathbb{R}^N$, $uR^*v \Leftrightarrow u'R^*v'$, if there exists $M \subset N$ such that $u_i = v_i$ and $u'_i = v'_i$ for each $i \in M$, whereas $u_i = u'_i$ and $v_i = v'_i$ for each $i \in N \setminus M$.

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Theorem: leximin

- **Theorem 4.16** (d'Aspremont and Gevers, 2002). A SWO R^*satisfying Strict Pareto*, Minimal Individual Symmetry*, Minimal equity*, and Inv*($\phi\left(u_i\right)$) is leximin.