ECON4310 Answers to Exercise 3

Due 19/9 2010

1 Ramsey model with log utility

(a)

$$\frac{c_{t+1}}{c_t} = \beta[1 + \alpha(k_{t+1})^{\alpha-1}] \quad t = 0, 1, 2, \ldots$$

(Note that the production function is $f(k) = k^{\alpha}$. Hence,$f'(k) = \alpha k^{\alpha-1}$).

(b) Consumption will be growing if, and only if, $\beta[1 + f'(k_{t+1})] = \beta[1 + (k_{t+1})^{\alpha-1}] > 1$. Equivalently, consumption will be growing if the the marginal productivity of capital $f'()$ is greater than the subjective discount rate ρ (where $\beta = 1/(1+\rho)$).

(c) The steady state (k_*, c_*) is defined as the values of k_t and c_t that make k and c stay constant. From the first-order condition and the constraint (2) $k_{t+1} = k_t = k_*$ and $c_{t+1} = c_t = c_*$ requires

$$\beta[1 + \alpha k_*^{\alpha-1}] = 1, \quad \text{or} \qquad \alpha k_*^{\alpha-1} = \rho \tag{3}$$

$$c_* = k_*^{\alpha} \tag{4}$$

The explanation for $f'(k)$ ending up strictly positive is impatience (subjective discounting, $\rho > 0$) .

(d) Saving is zero since net investment is zero.

(e) Note first that with $\beta = 0.96$ $\rho = (1-\beta)/\beta = 0.042$ From (1)

$$k_* = \left(\frac{\rho}{\alpha}\right)^{-1/(1-\alpha)} = \left(\frac{0.042}{0.3}\right)^{-1/(1-0.3)} = 16.6$$

From (2) we then find $c_* = 2.3$.

With $\beta = 0.98$ we find $k_* = 45.2, c_* = 3.13$.

2 Optimal growth with equal weight for all generations

(a) Focus on the terms in (3) that contain k_{t+1}:

$$U_0 = \ldots + \beta^t u(c_t A_t) + \beta^{t+1} u(c_{t+1} A_{t+1}) + \ldots$$

Take the derivative with respect to k_{t+1} and set it equal to zero to get the first order conditions:

$$\frac{\partial V}{\partial k_{t+1}} = \beta^t u'(c_t A_t) A_t \frac{dc_t}{dk_{t+1}} + \beta^{t+1} u'(c_{t+1} A_{t+1}) A_{t+1} \frac{dc_{t+1}}{dk_{t+1}} = 0$$

Replace dc_t/dk_{t+1} and dc_{t+1}/dk_{t+1} with the expressions you get from (4):

$$-\beta^t u'(c_t A_t)(1+n)(1+g)A_t + \beta^{t+1} u'(c_{t+1} A_{t+1}) A_{t+1} (1 + f'(k_{t+1}))$$

Divide through by $\beta^t A_{t+1}$ and change sides:

$$u'(c_t A_t)(1+n) = \beta u'(c_{t+1} A_{t+1})(1 + f'(k_{t+1})) = 0$$

Replace u' with the expressions we get from differentiating the CRRA utilty functions:

$$\frac{c_{t+1}}{c_t} = \left[\frac{1 + f'(k_{t+1})}{(1+\rho)(1+n)}\right]^\sigma \frac{1}{1+g}, \qquad t = 1, 2, \ldots$$

(b) Along a balanced growth path $c_{t+1} = c_t$ and $k_{t+1} = k_t = k_*$ The first order condition above then boils down to

$$\left[\frac{1 + f'(k_*)}{(1+\rho)(1+n)}\right]^\sigma \frac{1}{1+g} = 1$$

A log-linear approximation is

$$(i) \qquad f'(k_*) \approx \rho + n + \frac{1}{\sigma} g$$

This can be compared to what we get with strictly utilitarian preferences

$$(ii) \qquad f'(k_*) \approx \rho + \frac{1}{\sigma} g$$

and the golden rule

$$(iii) \qquad f'(k_*) \approx n + g$$

With $\sigma < 1$ and $\rho > n$ (which ensures that the maximization is meaningful in all cases) the required return on capital is highest in case (i), lowest in case (iii). The level of k_* is then highest in case capital stock in case (iii). If we compare (i) and (iii) the reasons for the higher return requirement and lower savings in the first case are two: *Impatience* ($\rho > 0$) and a *strong preference for equality* $\sigma < 1$). Both reduce savings, the latter because $g > 0$ is supposed to make later generations better off anyway. In the intermediate case the higher weight on the larger future generations dampens the effect of discounting.

(c) a) zero, b) g c) $g+n$

(d) Phase diagram as shown in the lectures, but with stationary point to the left. Starting point to the right of the stationary point. Curve for $k_{t+1} = k_t$ is given by

$$c_t = f(k_t) - (n + g + ng)k_t$$

The position of the vertical curve for $c_{t+1} = c_t$ is given in the answer to (b). Explain briefly how the starting point is pinned down by drawing one unsustainable and one inefficient path starting from different consumption levels but the same k.

(e) Curve for c constant shifts to the left, since the required return on capital increases (see (b)). Curve for k constant shifts down. New stationary point is down and to the left of the old.