ECON4310 Answers Exercise 4

Due 26/9 2011

CRRA utility and consumption choice over two periods

1. $u'(c) = c^{-\theta} > 0$ and $u''(c) = -\theta c^{-(1-\theta)} < 0$.
2. $u'(c) \to \infty$ when $c \to 0$, $u'(c) \to 0$ when $c \to \infty$.
3. A positive ρ means a preference for consumption in period 1 over consumption in period 2. If the consumer is asked to distribute a given quantity of goods between the two periods, he will prefer to allocate more than one half to period two, and more the higher is ρ. σ describes the willingness of the consumer to give up consumption in one period for consumption in the other period. A low substitution elasticity means that the compensation a consumer demands in terms of consumption in say period 1 for giving up consumption in say period 2 increases rapidly with the initial ratio between consumption in periods 1 and 2. Graphically a low σ means that the indifference curves are strongly curved. A low σ means an aversion to large differences in consumption between the two periods. [There are, of course, many ways to formulate the answer to this question].
4. First-order conditions can be found either by Lagrange's method or by using the budget equation to replace c_1 or c_2 in the utility function. It boils down to

   $$\frac{(1+\rho)c_1^{-\theta}}{c_2^{-\theta}} = 1 + r \Leftrightarrow \frac{c_2}{c_1} = \left(\frac{1+r}{1+\rho}\right)^{\sigma}$$

   The condition can be expressed in many different ways, but these two version are often chosen because of their interpretation, the first as saying that the marginal rate of substitution should be equal to the price ratio, the second saying that the growth rate of consumption depends on the relation between the market rate of interest and the consumer's degree of impatience (subjective discount rate).

   Her is an example of a detailed derivation: If we solve the budget constraint for c_2, we get:

   $$c_2 = (1 + r)(w - c_1)$$

Insert this in the utility function:

$$V = u(c_1) + \frac{1}{1+\rho}u((1+r)(w-c_1))$$

By differentiating with respect to c_1 we then get the first order condition

$$\frac{dV}{dc_1} = u'(c_1) - \frac{1}{1+\rho}u'((1+r)(w-c_1))(1+r) = 0$$

or, after moving the last term to the other side

$$u'(c_1) = \frac{1}{1+\rho}u'(c_2)(1+r)$$

Replace $u'(\cdot)$ by the derivative of the CRRA-function:

$$c_1^{-\theta} = \frac{1+r}{1+\rho}c_2^{-\theta}$$

Multiply by c_2^{θ} on both sides and you get

$$\left(\frac{c_2}{c_1}\right)^{\theta} = \frac{1+r}{1+\rho}$$

Solving this for c_2/c_1 yields

$$\frac{c_2}{c_1} = \left(\frac{1+r}{1+\rho}\right)^{1/\theta} = \left(\frac{1+r}{1+\rho}\right)^{\sigma}$$

5. Use the first-order condition to replace c_2 in the budget constraint:

$$c_1 + \frac{1}{1+r}\left(\frac{1+r}{1+\rho}\right)^{\sigma} c_1 = w$$

Solve this linear equation for c_1:

$$c_1 = \frac{(1+\rho)^{\sigma}}{(1+\rho)^{\sigma} + (1+r)^{\sigma-1}}w$$

Insert the solution for c_1 in the first order condition to find c_2.

$$c_2 = \frac{(1+r)^{\sigma}}{(1+\rho)^{\sigma} + (1+r)^{\sigma-1}}w$$

Demand for consumption in both periods is proportional to wage income. c_1 depends negatively on r if $\sigma > 1$, positively if $\sigma < 1$. Consumption in period 2 always depends positively on r, because the income and substitution effects work in the same direction. The easiest way to see this more formally is to rewrite the expression for c_2 as

$$c_2 = \frac{1}{(1+\rho)^{\sigma}(1+r)^{-\sigma} + (1+r)^{-1}}w$$

An increase in r reduces both terms in thee numerator and, hence, increases the fraction.

A higher ρ shifts consumption towards period 1 (increases c_1, reduces c_2).

Details on finding the solution for c_1 and c_2:

We start from (see above):

$$c_1 + \frac{1}{1+r}\left(\frac{1+r}{1+\rho}\right)^{\sigma} c_1 = w$$

Multiply with $(1+\rho)^{\sigma}$ on both sides:

$$c_1(1+\rho)^{\sigma} + c_1(1+r)^{\sigma-1} = w(1+\rho)^{\sigma}$$

Solve!

$$c_1 = \frac{(1+\rho)^{\sigma}}{(1+\rho)^{\sigma} + (1+r)^{\sigma-1}} w$$

From the first order condition:

$$c_2 = c_1\left(\frac{1+r}{1+\rho}\right)^{\sigma} = \frac{(1+\rho)^{\sigma}}{(1+\rho)^{\sigma} + (1+r)^{\sigma-1}} w$$

or,

$$c_2 = \frac{(1+r)^{\sigma}}{(1+\rho)^{\sigma} + (1+r)^{\sigma-1}} w$$