Investment: Tobin’s $q$
Lecture 10, ECON 4310

Tord Krogh

September 16, 2013
Today’s lecture is devoted to the theory of investment. Why? Because investment behavior is an important determinant for:

- Long run growth (through the role of capital)
- Business cycle dynamics
Investment over the business cycle

Investments are highly correlated with the business cycle and a lot more volatile than output. Here we have plots for the cyclical components of mainland GDP and oil investments for Norway (cyclical? Next lecture!). There are two lines because I have used two different measures of the cyclical component.

(a) Mainland GDP

(b) Oil investments
The investment components are by far the most volatile time series (again: cyclical components).

<table>
<thead>
<tr>
<th>Variable</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mainland GDP</td>
<td>2.49</td>
</tr>
<tr>
<td>Consumption</td>
<td>2.38</td>
</tr>
<tr>
<td>Public consumption</td>
<td>1.58</td>
</tr>
<tr>
<td>Investments except oil</td>
<td>9.96</td>
</tr>
<tr>
<td>Residential investment</td>
<td>10.48</td>
</tr>
<tr>
<td>Oil investment</td>
<td>14.77</td>
</tr>
<tr>
<td>Exports (except oil sector)</td>
<td>4.50</td>
</tr>
<tr>
<td>Imports (except oil sector)</td>
<td>5.57</td>
</tr>
</tbody>
</table>
So: What determines the rate of investment?

In the very first macro models you were taught, investment plays an important role over the business cycle. A typical Keynesian investment function is:

\[ I_t = a_0 + a_1 Y_t - a_2 r_t \]

where \( I_t \) is investment, \( Y_t \) is GDP and \( r_t \) is the real interest rate. So: Investment should be higher when the interest rate is low, and vice versa.
So: What determines the rate of investment? II

A similar view is found in typical central bankers’ view of how monetary policy affects inflation (illustration from Norges Bank’s website):
So: What determines the rate of investment? III

- However, ever since Haavelmo (1960) (*A Study in the Theory of Investment*), it has been recognized that the Keynesian investment function is inconsistent with the simple neoclassical model of investment.
- Moene and Rødseth (1991): Haavelmo’s aim in this book was to “destroy the standard Keynesian demand function for investment”, and “to offer an alternative”.
- Main point: Neoclassical theory only predicts a relationship between the desired stock of capital and the interest rate. No reason to expect a smooth relationship between investments and the interest rate.
Using a simple neoclassical model it is indeed possible to generate data series where the rate of investment is declining in the interest rate.

**Figure**: Scatter plot: Rate of investment as a function of the interest rate.
So: What determines the rate of investment? V

But the previous figure was just a coincidence for the first 20 observations. When I generate longer time series we see that the true relationship is much less smooth:

*Figure:* Scatter plot: Rate of investment as a function of the interest rate
The smooth plot was based on the first 20 observations where the interest rate was falling gradually from a high level and investment was rising slowly.

After that the interest rate varies around a trend and investment rises and falls without any clear relationship to the interest rate level.

**Figure:** Time series behind the scatter plots
Indeed, if we plot the rate of investment as a function of the change in the interest rate, it seems like we are much closer to an accurate description:

Figure: Scatter plot: Rate of investment as a function of the change in the interest rate
Outline

1. Neoclassical theory of investment

2. Capital adjustment costs: Tobin’s $q$

3. Tobin’s $q$ and the stock-market value

4. Summary
The value of a firm

Today we focus solely at firm’s investment behavior. But firms are owned by households, and (at least as a starting point) we should assume that firm managers maximize the value of the firm for its owners.

- Let $V_s$ denote the value of a firm at the end of period $s$
- Owners of the firm from the end of period $s$ receive dividends $d_{s+1}$ in the next period, and can sell the firm at a value $V_{s+1}$
- Assuming that there exists a risk free interest rate $r$, we know that the value in a perfect foresight case must satisfy:

$$1 + r = \frac{d_{s+1} + V_{s+1}}{V_s}$$
The value of a firm II

- Rewriting and then iterating forward we have:

\[ V_t = \sum_{s=t+1}^{T} \left( \frac{1}{1+r} \right)^{s-t} d_s + \left( \frac{1}{1+r} \right)^T V_{t+T} \]

- The no-bubble condition is \( \lim_{T \to \infty} \left( \frac{1}{1+r} \right)^T V_{t+T} = 0 \), and with that imposed we get:

\[ V_t = \sum_{s=t+1}^{\infty} \left( \frac{1}{1+r} \right)^{s-t} d_s \]

\( \Rightarrow \) The value of a firm reflects the NPV of future dividends.

- When making decisions, the firm manager should be concerned with maximizing the sum of dividends paid today plus the value at the end of period \( t \): \( V_t + d_t \), so the objective function should be:

\[ d_t + V_t = \sum_{s=t}^{\infty} \left( \frac{1}{1+r} \right)^{s-t} d_s \]
Firm behavior

Let's assume a production function \( A_t F(K_t, L_t) \). The firm hires labor but purchases its own capital (i.e. not rental market for capital). With no depreciation of capital, its profits in period \( t \) which are paid as dividends are:

\[
d_t = A_t F(K_t, L_t) - w_t L_t - [K_{t+1} - K_t]
\]

The firm manager thus chooses \( \{L_s, K_{s+1}\}_{s=t}^{\infty} \) to maximize:

\[
\sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} \{A_s F(K_s, L_s) - w_s L_s - [K_{s+1} - K_s]\}
\]

taking \( K_t \) as given. First-order conditions:

\[
A_s F'_L(K_s, L_s) = w_s
\]

for \( s \geq t \) and

\[
A_s F'_K(K_s, L_s) = r
\]

for \( s > t \).
How does this give us a theory of investment? Well, since \( I_t = K_{t+1} - K_t \), the rate of investment depends on what capital levels that come out of the first order conditions. Assume, for simplicity, that we are in a full employment equilibrium so \( L_s = \bar{L} \) for all \( s \). Then the first-order condition for labor just determines the real wage while

\[
A_s F'_{K}(K_s, \bar{L}) = r
\]

defines the optimal capital stock as a function of productivity and the interest rate, \( K^*(A_s, r) \). The rate of investment is then:

\[
I_s = K^*(A_{s+1}, r) - K^*(A_s, r)
\]
To see how the interest rate affects investment we need to see how $K^*$ changes:

$$\frac{dK^*}{dr} = \frac{r}{AF''_{KK}(K^*, L)}$$

which is negative under standard assumptions. We therefore have:

- Holding $K_t$ constant, $I_t$ will fall if $r$ goes up since $K_{t+1}$ falls
- But an increase in the interest rate level for all periods (or a change in the interest rate path) will affect both present and future capital! No reason to expect a smooth relationship between investment and the interest rate. Might be that investment can be both high and low for the same rate of interest.
- More important whether the interest rate is *rising* or *falling*, since that determines whether the capital stock is being *decreased* or *increased*.

This was one of the main points of Haavelmo (1960). Explains why the Keynesian investment function is inconsistent with a neoclassical theory of investment.
While this helps us understand the weakness of a Keynesian investment function, Haavelmo and others have pointed out that the basic neoclassical theory has a big problem as well.

Best understood in a continuous time setup. First-order condition for capital is then basically the same:

$$A(s)F'_K(K(s), \bar{L}) = r$$

while investment is $$I(s) = \frac{dK(s)}{ds} = \dot{K}(s)$$. By implicitly differentiating the first-order condition we find $$K^*$$ as a smooth function of $$r$$. So discrete changes in the interest rate must lead to discrete changes in the capital stock.

Discrete changes in the interest rate cause an infinite rate of investment!

This is another argument against the Keynesian function: Investment cannot be a smooth function of the interest rate.

But it is also a problem for the basic model: The rate of investment is clearly not infinite in practice.

In a discrete time model we of course don’t find infinite investment, but that is just camouflage.
Problem: Infinite investment II

- Put differently, the Keynesian investment function makes investment too smooth
- But the neoclassical model, although staring in the “right” place, predicts too rapid changes in investment.

⇒ Need to combine the neoclassical setup with a story for why investment is slower to adjust than in the baseline case

⇒ Natural solution: Add capital adjustment costs – which is what we will look at today

⇒ Haavelmo’s solution: Build a model where investors are rationed in booms and invest zero in recessions
Outline

1. Neoclassical theory of investment
2. Capital adjustment costs: Tobin's $q$
3. Tobin's $q$ and the stock-market value
4. Summary
Will now present a model with adjustment cost based on the presentation in Obstfeld and Rogoff (Chapter 2.5, 1996). Romer’s presentation is less suitable since it uses control theory. Advice for you: Learn the model as it is presented in this lecture, but use Romer’s discussion of the model for improved understanding. (Only Romer chapter 8.1-8.6 (3rd edition) that is on the syllabus).
In the baseline case the firm could purchase or sell capital with no cost other than the capital itself. Assume instead that capital is costly to adjust because of installation costs, production disruption, learning, etc. Profits in period $s$ are given by:

$$A_s F(K_s, L_s) - w_s L_s - I_s - \frac{\chi}{2} \frac{I_s^2}{K_s}$$

where $I_s = K_{s+1} - K_s$. Adjustment costs are convex, so it is costly to adjust a lot at the time. Why relative to capital? Both intuitive and convenient for the analytical results.
The firm now chooses capital and the rate of investment to maximize (we assume \( L_s = \bar{L} \) for simplicity):

\[
V_t = \sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} \left[ A_s F(K_s, \bar{L}) - w_s \bar{L} - I_s - \frac{\chi}{2} \frac{I_s^2}{K_s} \right]
\]

subject to \( K_{s+1} = K_s + I_s \). Let \( q_s \) be the current-valued Lagrange multiplier. Lagrangian expression:

\[
L_t = \sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} \left[ A_s F(K_s, \bar{L}) - w_s \bar{L} - I_s - \frac{\chi}{2} \frac{I_s^2}{K_s} - q_s (K_{s+1} - K_s - I_s) \right]
\]

First-order conditions:

\[
I_s : \quad - \chi \frac{I_s}{K_s} - 1 + q_s = 0
\]

\[
K_{s+1} : \quad - q_s + \frac{1}{1 + r} \left( A_{s+1} F_K(K_{s+1}, \bar{L}) + \frac{\chi}{2} \left( \frac{I_{s+1}}{K_{s+1}} \right)^2 + q_{s+1} \right) = 0
\]
The Lagrange multiplier $q$ plays a central role (yes, this is Tobin’s $q$). As any other Lagrange multiplier, it is a shadow price. In this case, $q_s$ is the shadow price of capital in place at the end of period $s$. From the first-order condition with respect to investment we see that:

$$q_s = 1 + \chi \frac{I_s}{K_s}$$

Under the optimal plan, the firm invests such that the marginal cost of an additional unit of capital (which equals 1 plus the adjustment cost) must equal the shadow price of capital. Can also write this as the investment equation that Tobin (1969) posited:

$$I_s = (q_s - 1) \frac{K_s}{\chi}$$

So investment is only positive when $q_s > 1$, i.e. when the shadow price of capital exceeds the price of new capital (before adjustments costs).
Next consider the first-order condition with respect to future capital.

\[ q_s = \frac{1}{1 + r} \left( A_{s+1} F_K(K_{s+1}, \bar{L}) + \frac{\chi}{2} \left( \frac{I_{s+1}}{K_{s+1}} \right)^2 + q_{s+1} \right) \]

This is like an investment Euler condition. The shadow price of capital today must equal the discounted value of

- the return of capital next period,
- what you save in adjustment costs next period
- the future shadow price (since capital can be sold next period).
In the same way as we used iterative substitution to rewrite the value of a firm, we can impose \( \lim_{T \to \infty} \frac{q_t + T}{(1+r)^T} = 0 \) (i.e. no bubble) and get:

\[
q_t = \sum_{s=t+1}^{\infty} \left( \frac{1}{1+r} \right)^{s-t} \left[ A_s F_K(K_s, \bar{L}) + \frac{\chi}{2} \left( \frac{l_s}{K_s} \right)^2 \right]
\]

so \( q_t \) reflects the NPV of all future marginal return and reduced adjustment cost that you get from purchasing one unit of capital.
OK: We have two first-order conditions as well as the constraint $l_t = K_{t+1} - K_t$. How to proceed? Use the first order condition for $l_t$ to insert for investment in the two other equations. What we are left with is:

$$\Delta K_{t+1} = (q_t - 1) \frac{K_t}{\chi}$$

(1)

$$\Delta q_{t+1} = r q_t - A F_K(K_t(1 + \frac{q_t - 1}{\chi}), \bar{L}) - \frac{1}{2\chi}(q_{t+1} - 1)^2$$

(2)

which we can use to draw a phase diagram for the dynamics of $K$ and $q$. To do so we need:

- To know the steady state
- And then describe how $K$ and $q$ evolve away from steady state
Phase diagram II

The steady state is the case with $\Delta q = \Delta K = 0$:

$$0 = (\bar{q} - 1) \frac{\bar{K}}{\chi}$$

$$0 = r\bar{q} - AF_K(\bar{K}(1 + \frac{\bar{q} - 1}{\chi}), \bar{L}) - \frac{1}{2\chi}(\bar{q} - 1)^2$$

From the first equation it is clear that $\bar{q} = 1$. From the second we get that $\bar{K}$ must satisfy $AF_K(\bar{K}, \bar{L}) = r$. 
Then we can use (1) and (2) to describe the dynamics away from steady state. However, the analysis is somewhat easier if we study a linear approximation of the system close to the steady state. The first-order approximation of (1) is:

$$\Delta K_{t+1} = (q_t - 1) \frac{\bar{K}}{\chi} \quad (3)$$

while from (2) we get:

$$\Delta q_{t+1} = \left( r - \frac{A\bar{K}F_{KK}(\bar{K}, \bar{L})}{\chi} \right) (q_t - 1) - AF_{KK}(\bar{K}, \bar{L})(K_t - \bar{K}) \quad (4)$$
Setting $\Delta K_{t+1} = 0$ in (3) a horizontal schedule at $q_t = 1$. For $q_t > 1$, the capital stock is growing and for $q_t < 1$ it is decreasing (because you invest only when $q > 1$).
Phase diagram V

Setting $\Delta q_{t+1} = 0$ in (4) a downward sloping schedule for which $(K, q)$ combinations that keep Tobin’s $q$ constant. For capital stocks to the left of the schedule the return to capital is high, so the price today grows relative to the future price (i.e. $\Delta q < 0$), so $q$ is falling. To the right of the schedule the return on capital is so low that we need large capital gains to satisfy the optimality condition. $q$ must grow.
Phase diagram VI

This system features saddle-path stability: For a given $K_0$ it is a unique level of $q$ that puts the firm on a path that converges at the steady state.
Starting from $K_0$, the firms capital stock is low and its return to capital is high.

Going straight to the steady state capital stock (as would have happened without adjustment costs) is too costly.

From (2) it is clear that such a situation leads to a high (above 1) $q_t$ (and $\Delta q_{t+1} < 0$) since an extra unit of capital in this firm is valuable.

The high shadow value stimulates investment, so $K$ grows.

As the capital stock is increasing, the shadow value falls, which then makes investment fall. In the end it converges to zero.

So having adjustment costs makes it possible to produce a more realistic investment response in which (i) the rate of investment is not infinite (even in continuous time), and (ii) that recognizes that it takes time to adjust the capital stock.
Phase diagram VIII

The unstable paths can be ruled out by assuming \( \lim_{T \to \infty} (1 + r)^{-T} q_{t+T} = 0 \). In addition the one going to the south-west quadrant gives \( q < 0 \) and \( K < 0 \) at some point – which is not possible. Both paths are examples of bubbles; in the one case capital is increasing solely due to ever-increasing beliefs of \( q \), while the other represents ever-decreasing beliefs of \( q \).
What happened to the interest rate? Recall that we were discussing the relationship between the interest rate and investment at the start of the lecture. We will now do three different experiments:

- A permanent reduction in $r$ from $r_0$ to $r_1$
- A reduction in $r$ believed to be permanent from $r_0$ to $r_1$ and then an increase to a level between $r_1$ and $r_0$
- An increase in $r$ that is anticipated in advance
First experiment: A permanent reduction in the interest rate. This affects the $\Delta q = 0$-locus. When the interest rate falls, a higher level of $q$ is needed to keep $\Delta q = 0$, so the curve shifts up. There is a new saddle path. A jump in $q$ makes us move from the old steady state (A) to the new saddle path (B). Will slowly converge to new steady state (C) with a higher capital stock.
Investment and the interest rate III

Second experiment: The same reduction as in experiment 1. But when the firm has come to point D, the interest rate is suddenly raised again to a level between the first and the second. Yet another saddle path, and we jump from (D) to (E). For this interest rate the capital stock is too large, so we get a period of disinvestment until we converge at (F).
Experiment 1 shows that the model retains the intuitively plausible mechanism that a lower interest rate leads to higher investment. Further, it leads to a higher value of the firm, which is also intuitive.

However, as experiment 2 shows, the relationship between investment and the interest rate is still far from as smooth as in the Keynesian investment function.

⇒ Point from Haavelmo still holds: There is no relationship between the rate of interest and the rate of investment.

We see that the movement from (E) to (F) gives disinvestment even though we can observe positive investment for a higher interest rate if we start out with \( K < \bar{K} \).
Third experiment: This time the firm anticipates a higher interest rate in the future. What happens? We know two things:

- A higher interest rate gives a new $\Delta q = 0$-locus and therefore also a new saddle path. When $r$ changes, the new $(K, q)$-combination must be on the saddle path
- But we also know that (in this case perfect foresight but more general) a forward looking firm will not wait until $r$ changes: it is only with the arrival of news that $q$ jumps

This means that $q$ will jump before the interest rate moves. Then it will be on a smooth path that ends up at the new saddle path exactly when $r$ increases.
Investment and the interest rate VI

This is seen from the figure where we jump from (A) to (B) as soon as the news arrive. But since \( r \) hasn’t changed yet the dynamics are still governed by the old system so we are on an unstable path. This takes us to point (C) exactly at the point in time when \( r \) increases. Afterwards we converge to \( D \). Note that the value \( q \) “overshoots”. Note also that the further ahead the change is expected, the smaller is the initial jump.
Outline

1. Neoclassical theory of investment

2. Capital adjustment costs: Tobin’s $q$

3. Tobin’s $q$ and the stock-market value

4. Summary
Average and marginal $q$

- $q_t$ is the shadow price of capital held at the end of period $t$ in the firm, so $q_t K_{t+1}$ is a natural way to value the firm. But $q_t$ is unobservable.

- We have already derived the value of the firm $V_t$ as the NPV of future dividends from the firm. This is presumably the stock-market value of the firm.

- Is there a relation between $q_t K_{t+1}$ and $V_t$?
Average and marginal $q$ II

Assume first that there are no capital adjustment costs. Then $q_t = 1$.

- Since it is capital that makes up the firm, one should expect that the value of the firm equals its capital stock $K_t$.
- But we also have that $V_t$ equals the NPV of future dividends.
- These valuations are consistent since:

$$AF(K, L) = AF_K K + AF_L L = rK + wL$$

(Euler’s theorem), so we can write dividends as:

$$d_s = [r_s K_s + w_s L_s] - w_s L_s - [K_{s+1} - K_s] = (1 + r)K_s - K_{s+1}$$

which makes the value of the firm:

$$V_t = \sum_{s=t+1}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} [(1 + r)K_s - K_{s+1}] = K_{t+1}$$
Tobin’s $q$ and the stock-market value

**Average and marginal $q$ III**

Then we have adjustment costs. We start with (2):

$$q_t = \frac{1}{1 + r} \left( A_{t+1} F_K (K_{t+1}, \bar{L}) + \frac{\chi}{2} \left( \frac{l_{t+1}}{K_{t+1}} \right)^2 + q_{t+1} \right)$$

Multiply both sides by $K_{t+1}$ and use $K_{t+1} = K_{t+2} - I_{t+1}$ for the last term to the right:

$$q_t K_{t+1} = \frac{1}{1 + r} \left( A_{t+1} F_K (K_{t+1}, \bar{L}) K_{t+1} + \frac{\chi}{2} \frac{l_{t+1}^2}{K_{t+1}} - q_{t+1} I_{t+1} + q_{t+1} K_{t+2} \right)$$

Insert for $q_{t+1} = 1 + \chi (I_{t+1}/K_{t+1})$:

$$q_t K_{t+1} = \frac{1}{1 + r} \left( A_{t+1} F_K (K_{t+1}, \bar{L}) K_{t+1} - \frac{\chi}{2} \frac{l_{t+1}^2}{K_{t+1}} - I_{t+1} + q_{t+1} K_{t+2} \right)$$

Do a forward iteration and impose $\lim_{T \to \infty} (1 + r)^{-T} q_{t+T} K_{t+T+1} = 0$:

$$q_t K_{t+1} = \sum_{s=t+1}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} \left( A_s F_K (K_s, \bar{L}) K_s - \frac{\chi}{2} \frac{l_s^2}{K_s} - l_s \right)$$
Finally use that the production function is homogeneous of degree one so

\[ A_s F_K(K_s, \bar{L}) = A_s F(K_s, \bar{L}) - w_s \bar{L} \]

which means:

\[ q_t K_{t+1} = \sum_{s=t+1}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} \left( A_s F(K_s, \bar{L}) - w_s \bar{L} - \frac{\chi}{2} \frac{I_s^2}{K_s} - I_s \right) = V_t \]

Perfect! So \( q_t K_{t+1} \) equals the stock market value of the firm.
Average and marginal $q$ $V$

- $q_t$ is often referred to as *marginal* $q$ and is unobservable.
- $V_t/K_{t+1}$ is the *average* $q$ and can be measured.
- But unfortunately only under a set of simplifying assumptions that average and marginal $q$ coincide (see Hayashi (1982)).
Outline

1. Neoclassical theory of investment
2. Capital adjustment costs: Tobin’s $q$
3. Tobin’s $q$ and the stock-market value
4. Summary
Summary

- Keynesian investment function $I(r)$ is too naive
- Simple neoclassical model shows that there is no simple relationship between investment and the rate of investment
- But the simple neoclassical model makes the capital stock adjust too quickly
- Capital adjustment costs makes the response more realistic
- But Haavelmo’s critique of the Keynesian investment function continues to hold
- Tobin’s model not only makes investment more realistic: it also gives us a link between investment and the shadow value of capital in the firm $q$
- Under simplifying assumptions, $q$ can be identified from the stock market value per unit of capital