1 Lecture Notes: The Solow Model

Questions:
1. Why large differences in growth rates?
2. Why persistent differences in productivity?
3. What drives overall world growth?

1.1 The Solow model: theory

- Technology: Firms produce a generic good $Y$ with the production function

$$Y = F(K, AL) = K^\alpha (AL)^{1-\alpha}$$

- $L$ is the number of workers
- $A$ is the number of "efficiency units" per worker
- $AL$ is the aggregate number of efficiency units. Assume implicitly that all efficiency units are perfect substitutes)
- Note that $F$ satisfies constant return to scale and the Inada conditions
- The generic good can be used for consumption and investment
- Assume constant growth in $L$ and $A$:

$$L_{t+1} = (1 + n) L_t$$
$$A_{t+1} = (1 + g) A_t$$

- Law of motion for capital:

$$K_{t+1} = (1 - \delta) K_t + I_t$$

- Closed economy

- Preferences: Assume a labor supply of one unit per person and assume a constant savings rate:

$$S_t = s Y_t$$

- Markets: Competitive markets for labor, capital, and the consumption/investment good.

- Competitive equilibrium:
1. Firm optimization: prices are marginal productivities,
\[
\max_{K,L} \{ F(K,AL) + (1 - \delta)K - wL - (1 + r)K \} \\
\Rightarrow \\
0 = F_1(K,AL) + 1 - \delta - (1 + r) \\
0 = F_2(K,AL) - w \\
\Rightarrow \\
r + \delta = \alpha K^{\alpha-1} (AL)^{1-\alpha} = \alpha \frac{Y}{K} \\
w = \alpha K^{\alpha} (AL)^{1-\alpha} / L = \alpha \frac{Y}{L}
\]

2. Individual optimization: CHEAT by assuming \(l = 1\) and \(C = (1 - s)Y\).

3. Market clearing
   (a) Market for labor:
   (b) Market for capital: Closed economy implies savings equal investments
\[ S_t = I_t \]
   (c) Goods market clearing requires
\[ Y_t = C_t + I_t \]

- Normalize capital and output per efficiency unit
\[ y_t = \frac{Y_t}{A_t L_t} \]
\[ k_t = \frac{K_t}{A_t L_t} \]

Note that
\[ y_t = \frac{K_t^\alpha (A_t L_t)^{1-\alpha}}{A_t L_t} = \left( \frac{K_t}{A_t L_t} \right)^\alpha = k_t^\alpha \equiv f(k_t) \]

- Dynamics of the economy: Note: \(K_{t+1}\) is a stock, while \(I_t\) and capital growth \(\Delta K_{t+1} = K_{t+1} - K_t\) are flows
\[ \Delta K_{t+1} = K_{t+1} - K_t = (1 - \delta)K_t + I_t - K_t \]
\[ = sY_t - \delta K_t \]
In efficiency units:

\[
\Delta k_{t+1} = \frac{k_{t+1} - k_t}{A_{t+1}L_{t+1} - A_tL_t} = \frac{k_{t+1}}{A_{t+1}L_{t+1}} - \frac{k_t}{A_tL_t} = \frac{K_{t+1}}{A_{t+1}L_{t+1}} - \frac{K_t}{A_tL_t} (1 + g) (1 + n)
\]

\[
= \frac{K_{t+1} - K_t - K_t [(1 + g) (1 + n) - 1]}{A_{t+1}L_{t+1}} = \frac{sY_t - K_t [g + n + \delta + gn]}{A_{t+1}L_{t+1}}
\]

\[
\Delta k_{t+1} \approx sf(k_t) - \frac{[g + n + \delta + gn] k_t}{\text{gross inv.}} \quad \text{replacement needed to keep capital constant}
\]

- Definition: a steady state (with technical progress and population growth) is an equilibrium path in which \(\Delta k_{t+1} = 0\).

- Solve for steady state \(k^*\):

\[
0 = sf(k^*) - [g + n + \delta + gn] k^*
\]

\[
\Rightarrow \frac{f(k^*)}{k^*} = (k^*)^\alpha = \frac{g + n + \delta + gn}{s} \Rightarrow k^* = \left(\frac{s}{g + n + \delta + gn}\right)^{1/\alpha}
\]

- Consider effect on \(k^*\) from changes in savings rate \(s\) and in population growth \(n\). GRAPH \((y, k\) graph and time-series\)
• **Golden rule**: solve for the maximum steady-state consumption \( c^{**} \):

\[
C^* = Y^* - I^* \\
I^* = (g + n + \delta + gn) ALk^* \\
c^* = \frac{Y^* - I^*}{AL} = \frac{Y^* - (g + n + \delta + gn) ALk^*}{AL} \\
\approx f(k^*) - (g + n + \delta) k^* \\
\]

\[
\frac{\partial c^*}{\partial k^*} = f'(k^{**}) - (g + n + \delta) = 0 \\
\Rightarrow f'(k^{**}) = g + n + \delta
\]

Recall that, from firm’s optimization,

\[
r_t + \delta = f'(k_t)
\]

so requirement for **golden rule** is

\[
r^{**} = n + g,
\]

i.e., all return to capital is reinvested (zero dividends consumed by capitalists).

### 1.2 Empirical performance of the Solow model

- **Want to evaluate the Solow model empirically. Focus on dynamics (i.e., speed of growth).**

- "Neoclassical hypothesis": all differences in growth and GDP per capita are due to differences in capital

- **Problem**: difficult to measure capital. Solution: derive implied growth in \( Y_t \) without need to measure \( K_t \):

\[
Y = K^\alpha (AL)^{1-\alpha} \\
\Rightarrow K = Y^{\frac{1}{\alpha}} (AL)^{-\frac{1-\alpha}{\alpha}} \\
K_{t+1} = sY_t + (1 - \delta) K_t \\
Y_{t+1}^{\frac{1}{\alpha}} (A_{t+1}L_{t+1})^{-\frac{1-\alpha}{\alpha}} = sY_t + (1 - \delta) Y_t^{\frac{1}{\alpha}} (A_tL_t)^{-\frac{1-\alpha}{\alpha}} \\
\left(\frac{Y_{t+1}}{L_{t+1}}\right)^{\frac{1}{\alpha}} (A_{t+1})^{-\frac{1-\alpha}{\alpha}} L_{t+1} = s \left(\frac{Y_t}{L_t}\right) L_t + (1 - \delta) \left(\frac{Y_t}{L_t}\right)^{\frac{1}{\alpha}} (A_t)^{-\frac{1-\alpha}{\alpha}} L_t \\
\left(\frac{A_{t+1}}{A_t}\right)^{-\frac{1-\alpha}{\alpha}} \frac{L_{t+1}}{L_t} \left(\frac{Y_{t+1}}{L_{t+1}}\right)^{\frac{1}{\alpha}} = (A_t)^{-\frac{1-\alpha}{\alpha}} s \left(\frac{Y_t}{L_t}\right) + (1 - \delta) \left(\frac{Y_t}{L_t}\right)^{\frac{1}{\alpha}}
which boils down to

\[(1 + n) (1 + g) \frac{1-\alpha}{\alpha} \left(\tilde{y}_{t+1}\right)^{\frac{1}{\alpha}} = \left(A_t\right)^{\frac{1-\alpha}{\alpha}} s \tilde{y}_t + (1 - \delta) \left(\tilde{y}_t\right)^{\frac{1}{\alpha}} \tag{1}\]

where \(\tilde{y}\) is GDP per capita:

\[\tilde{y}_t = \frac{Y_t}{L_t}\]

1. Assume that all countries have the same \(\delta = 5\%\) and \(g = 2.5\%\) (same as the US, 1960-2000). Note: \(1 - \alpha\) is labor’s share of output: from firm’s optimization we have

\[w = (1 - \alpha) \frac{Y}{L} \Rightarrow 1 - \alpha = \frac{wL}{Y},\]

which is 2/3 for the US \(\Rightarrow \alpha = 1/3\).

2. Assume that the US is in steady state in 1960. Measure \(Y_{US}^{1960}\) and \(L_{US}^{1960}\). Pin down \(K_{US}^{1960}\) and \(A_{US}^{1960}\)

3. Measure \(\tilde{y}_{j1960} = Y_{j1960}^{1960}/L_{j1960}^{1960}\) for a country \(j\).

4. Assume that \(A_{1960}^j = A_{US}^{1960}\). Use equation (1) to project future values of \(Y_{j1960}^t\) (note: for the US it is simply

\[\tilde{y}_{US}^t = (1 + g)^{t-1960} \cdot \tilde{y}_{US}^{1960}\]

5. Observation: Solow model implies too fast convergence. So a large amount of the differences in output across countries must be driven by differences in \(A_{j1960}^t\).

1.3 Conclusion

1. Empirical standpoint: Solow model fails to explain in a satisfactory way the great disparities in output levels and growth rates

2. Theoretical standpoint: Solow model (or simple extensions of it) cannot explain the growth in \(A_t\), which is the main drive of growth

Need "new growth theory" to explain why \(A_{j1960}^t\) does or does not grow.