# Problem set 1 

## ECON 4330

We are looking at an open economy that exists for two periods. Output in each period $Y_{1}$ and $Y_{2}$ respectively, is given exogenously. A representative consumer maximizes life-time utility

$$
U=u\left(C_{1}\right)+\beta u\left(C_{2}\right)
$$

where $C_{1}$ and $C_{2}$ are consumption in the two periods and $\beta$ is a subjective discount factor, $0<\beta<1$. The country can borrow and lend in world markets at a given real interest rate, $r$. The initial asset is zero. Hence, the budget constraint can be written

$$
C_{1}+\frac{C_{2}}{1+r}=Y_{1}+\frac{Y_{2}}{1+r}
$$

1. Derive the first order condition for optimal consumption and interpret it.
solution Using the method of substitution (if you want, use Lagrange method). Solve the budget constraint for $C_{2}$

$$
\begin{equation*}
C_{2}=Y_{1}(1+r)+Y_{2}-(1+r) C_{1} \tag{1}
\end{equation*}
$$

and substitute the constraint into the objective function

$$
\begin{equation*}
U=u\left(C_{1}\right)+\beta u\left(Y_{1}(1+r)+Y_{2}-(1+r) C_{1}\right) \tag{2}
\end{equation*}
$$

Maximize the objective function wrt $C_{1}$. The FOC is

$$
\begin{align*}
u^{\prime}\left(C_{1}\right)-\beta u^{\prime}\left(C_{2}\right)(1+r) & =0 \\
& \Leftrightarrow \\
u^{\prime}\left(C_{1}\right) \frac{1}{1+r} & =\beta u^{\prime}\left(C_{2}\right) \\
& \Leftrightarrow \\
\frac{\beta u^{\prime}\left(C_{2}\right)}{u^{\prime}\left(C_{1}\right)} & =\frac{1}{1+r} \tag{3}
\end{align*}
$$

2. Derive the welfare effects of an increase in $r$, i.e $d U / d r$. Provide intuition. (use the envelope condition)
solution Substituting 1 into 3 , gives us one equataion to the determine optimal period 1 consumption $C_{1}^{*}$ as function of the exogenous variables $C_{1}^{*}\left(r, Y_{1}, Y_{2}\right)$. Optimal period two consumption follows from eq.

1: $C_{2}^{*}\left(r, Y_{1}, Y_{2}\right)=(1+r) Y_{1}+Y_{2}-(1+r) C_{1}^{*}\left(r, Y_{1}, Y_{2}\right)$. Substituting this into the objective function gives the indirect utility function, i.e. maximal utility as a function of exogenous variables
$U^{*}\left(r, Y_{1}, Y_{2}\right)=u\left(C_{1}^{*}\left(r, Y_{1}, Y_{2}\right)\right)+\beta u\left(Y_{1}(1+r)+Y_{2}-(1+r) C_{1}^{*}\left(r, Y_{1}, Y_{2}\right)\right)$
To see how welfare is affected by an increase in the interest rate we derive

$$
\begin{aligned}
\partial U^{*} / \partial r & =u^{\prime}\left(C_{1}^{*}\right) \frac{\partial C_{1}^{*}}{\partial r}+\beta u^{\prime}\left(C_{2}^{*}\right)\left[Y_{1}-C_{1}^{*}-(1+r) \frac{\partial C_{1}^{*}}{\partial r}\right] \\
& =\beta u^{\prime}\left(C_{2}^{*}\right)\left(Y_{1}-C_{1}^{*}\right)+\frac{\partial C_{1}^{*}}{\partial r}\left[u^{\prime}\left(C_{1}^{*}\right)-(1+r) \beta u^{\prime}\left(C_{2}^{*}\right)\right]
\end{aligned}
$$

In optimum the FOC holds, $u^{\prime}\left(C_{1}^{*}\right)-(1+r) \beta u^{\prime}\left(C_{2}^{*}\right)=0$, and we are left with

$$
\partial U^{*} / \partial r=\beta u^{\prime}\left(C_{2}^{*}\right)\left(Y_{1}-C_{1}^{*}\right)
$$

3. Assume CRRA utility

$$
u(C)=\frac{C^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}
$$

Find an expression for date 1 consumption and current account as functions of exogenous variables $\left(Y_{1}, Y_{2}, r\right)$.
4. Assume $Y_{2}=0$
(a) Derive $\partial C_{1} / \partial r$ and find condition that makes sure the current account is improving when the world interest rate goes up.
solution Current account in a model without investment $C A_{1}=S_{1}=$ $Y_{1}-C_{1}$. To find the consumption response to $r$ combine the FOC

$$
C_{2}=(\beta(1+r))^{\sigma} C_{1}
$$

and the budget constraint

$$
\begin{aligned}
C_{1}+\frac{C_{2}}{1+r} & =Y_{1} \\
C_{1}\left(1+\beta^{\sigma}(1+r)^{\sigma-1}\right) & =Y_{1} \\
C_{1} & =\frac{1}{\left(1+\beta^{\sigma}(1+r)^{\sigma-1}\right)} Y_{1}
\end{aligned}
$$

take the derivative wrt $r$ to get

$$
\frac{\partial C_{1}}{\partial r}=-\frac{(\sigma-1) \beta^{\sigma}(1+r)^{\sigma-2}}{\left(1+\beta^{\sigma}(1+r)^{\sigma-1}\right)^{2}} Y_{1}
$$

If $\sigma>1$ the date 1 consumption goes down and saving increases.
(b) Describe how $C_{1}$ responds to changes in $r$ in terms of substitution and income effects.
solution As in seminar.
(c) What additional effect comes in if we assume $Y_{2}>0$ ?
solution with a positive $Y_{2}$ a decrease in $p_{2}$ reduces the value of period 2 endowment. Hence, there is an additional negative wealth effect that reduces both $C_{1}$ and $C_{2}$. Note that the income effect is given by

$$
\frac{\partial C_{1}}{\partial I} C_{2}
$$

where $I=Y_{1}+p_{2} Y_{2}$. The wealth effect is given by

$$
-\frac{\partial C_{1}}{\partial I} \frac{\partial I}{\partial p_{2}}=-\frac{\partial C_{1}}{\partial I} Y_{2}
$$

Hence the combined income and wealth effect is

$$
\left(C_{2}-Y_{2}\right) \frac{\partial C_{1}}{\partial I}
$$

5. Suppose a foreign country has the same preferences as the home country, equal date 1 output $Y_{1}^{*}=Y_{1}$ but different date 2 output $Y_{2}^{*}$
(a) Assume higher income growth in the home country, i.e. $Y_{2}>Y_{2}^{*}$. Derive the autarky interest rate in both countries and compare.
solution The autarky interest rate $r_{A}$ is the interest rate that induces the agent to set consumption equal to income in both periods.

$$
\frac{\beta C_{2}^{-\frac{1}{\sigma}}}{C_{1}^{-\frac{1}{\sigma}}}=\frac{1}{1+r_{A}}
$$

We find the autarky interest rate by inserting for the only possible equilibrium allocation when there's no trade $C_{t}=Y_{t}$ and solve for the interest rate

$$
\begin{aligned}
1+r_{A} & =\beta\left(\frac{Y_{2}}{Y_{1}}\right)^{\frac{1}{\sigma}} \\
1+r_{A}^{*} & =\beta\left(\frac{Y_{2}^{*}}{Y_{1}^{*}}\right)^{\frac{1}{\sigma}} \\
r_{A}^{*} & <r_{A}
\end{aligned}
$$

extra Note that the autarky interest rate is bounded below at -1

$$
r_{A}=\beta\left(\frac{Y_{2}}{Y_{1}}\right)^{\frac{1}{\sigma}}-1
$$

since $\beta\left(\frac{Y_{2}}{Y_{1}}\right)^{\frac{1}{\sigma}}>0$.
(b) Suppose the world market consists of these two countries. State the equilibrium condition, and show in a graph how the interest rate will be determined. Which country will run a current account surplus in period 1? Intuitively, what are the gains from trade?
solution Since there is no capital in this model, the world saving in period $1 S_{1}^{w}$ has to be zero, i.e.

$$
S^{w}=S+S^{*}=0
$$

where $S=Y_{1}-C_{1}$ and $S^{*}=Y_{1}^{*}-C_{1}^{*}$. Note that in a model without capital, the current account equals saving $C A_{1}=S_{1}$. The home and foreign saving depends on the interest rate. The equilibrium interest rate $r$ is the rate that makes

$$
S(r)=-S^{*}(r)
$$

Home country will run a current account deficit and Foregin a surplus.
(c) Using the answer from question 2. what happens to welfare in the home country if the foreign country's output growth increases $\left(Y_{2}^{*}\right.$ up)?
solution If $Y_{2}^{*}$ increases $r_{A}^{*}$ goes up and $r$ goes up. Since Home is a borrower in period 1 its welfare goes down (worsening of terms of trade).

## Part 2

In this problem we consider an infinite horizon model with a representative agent and perfect foresight. Each period, the agent must obey the following budget constraint:

$$
C_{s}+B_{s+1}=Y_{s}+(1+r) B_{s}
$$

1. Based on the fact that the budget constraint holds for every period from $t$ to $t+T$, show that this implies

$$
(1+r) B_{t}=\sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t}\left(C_{s}-Y_{s}\right)+\frac{B_{t+T+1}}{(1+r)^{T}}
$$

Solution Since the period-by-period budget constraint holds for every period $t$ to $t+T$ we have

$$
\begin{align*}
B_{t}= & \frac{1}{1+r}\left(C_{t}-Y_{t}+B_{t+1}\right) \\
B_{t+1}= & \frac{1}{1+r}\left(C_{t+1}-Y_{t+1}+B_{t+2}\right)  \tag{4}\\
B_{t+2}= & \frac{1}{1+r}\left(C_{t+2}-Y_{t+2}+B_{t+3}\right)  \tag{5}\\
& \cdots  \tag{6}\\
B_{t+T}= & \frac{1}{1+r}\left(C_{t+T}-Y_{t+T}+B_{t+T+1}\right)
\end{align*}
$$

Using these constraints, start by re-writing the period $t$ budget constraint

$$
(1+r) B_{t}=C_{t}-Y_{t}+B_{t+1}
$$

Inserting for $B_{t+1}$ using eq 4

$$
(1+r) B_{t}=C_{t}-Y_{t}+\underbrace{\frac{1}{1+r}\left(C_{t+1}-Y_{t+1}+B_{t+2}\right)}_{B_{t+1}}
$$

Continue by substitute for $B_{t+2}$ using eq 5

$$
\begin{aligned}
(1+r) B_{t} & =C_{t}-Y_{t}+\underbrace{\frac{1}{1+r}[C_{t+1}-Y_{t+1}+\underbrace{\frac{1}{1+r}}_{B_{t+2}}\left(C_{t+2}-Y_{t+2}+B_{t+3}\right)}_{B_{t+1}}] \\
& =C_{t}-Y_{t}+\frac{C_{t+1}-Y_{t+1}}{1+r}+\frac{C_{t+2}-Y_{t+2}}{(1+r)^{2}}+\frac{B_{t+3}}{(1+r)^{2}}
\end{aligned}
$$

and continue up to period $T$.
2. Explain the intuition behind

$$
\lim _{T \rightarrow \infty} \frac{B_{t+T+1}}{(1+r)^{T}}=0
$$

solution As in seminar.
3. Impose this restriction and assume that $C_{s}=c Y_{s}$ and $Y_{s}=(1+g)^{s-t} Y_{t}$.
(a) Find the intertemporal budget constraint for this case (when $g<r$ ).
solution The intertemporal budget constraint (aka life-time budget constraint) is found by letting $T \rightarrow \infty$ and imposing the restriction $\lim _{T \rightarrow \infty} \frac{B_{t+T+1}}{(1+r)^{T}}=0$

$$
(1+r) B_{t}=\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}\left(C_{s}-Y_{s}\right)
$$

if we further assume that $C_{s}=c Y_{s}$ and $Y_{s}=(1+g)^{s} Y_{t}$ we get

$$
\begin{aligned}
(1+r) B_{t} & =Y_{t}(c-1) \sum_{s=t}^{\infty}\left(\frac{1+g}{1+r}\right)^{s-t} \\
(1+r) B_{t} & =Y_{t}(c-1) \frac{1+r}{r-g} \\
\frac{c-1}{r-g} & =\frac{B_{t}}{Y_{t}}
\end{aligned}
$$

where we have used that the infinite sum $\sum_{s=t}^{\infty}\left(\frac{1+g}{1+r}\right)^{s-t}$ converges when $g<r(\Rightarrow(1+g) /(1+r)<1)$.
(b) Imagine that keeping consumption at a fixed share $c$ of output indeed is the optimal consumption-choice of a representative agent. Is $c$ above or below one?
solution Use the intertemporal budget constraint to solve for the maximal sustainable level of $c$.

$$
c=(r-g) \frac{B_{t}}{Y_{t}}+1
$$

If the agent initially has positive assets $B_{t}>0$ then $c>1$. If the agent initially has debt $B_{t}<0$ then $c<1$
(c) Assume $g=0$. What does the time-profile of $B_{t}$ look like for a given value of $c$ ?
solution if $g=0$ then $Y_{s}=Y_{t}=Y$ is constant. We get

$$
c=r \frac{B_{t}}{Y}+1
$$

So

$$
C=c Y=r B_{t}+Y
$$

Hence the agent consumes the interest on the asset each period and the $B_{t}$ stays constant. The evolution of $B_{t+1}$ is given from the period-by-period budget constraint

$$
\begin{aligned}
B_{t+1} & =Y-C+(1+r) B_{t} \\
& =Y-\left(r B_{t}+Y\right)+(1+r) B_{t} \\
& =B_{t}
\end{aligned}
$$

and so on. Hence, $B_{t}$ is constant. This amounts to consuming income $Y$ plus (minus) interest payment on initial asset (debt) $r B_{t}$ each period.
(d) Assume $g>0$ (but also $g<r$ ). What does the time-profile look like now?
solution The solution for $c$ is now

$$
c=(r-g) \frac{B_{t}}{Y_{t}}+1
$$

$B_{t+1}$ is given by

$$
\begin{aligned}
B_{t+1} & =Y_{t}-C_{t}+(1+r) B_{t} \\
& =Y_{t}(1-c)+(1+r) B_{t} \\
& =(g-r) B_{t}+(1+r) B_{t} \\
& =(1+g) B_{t}
\end{aligned}
$$

$B_{t+2}$ is given by

$$
\begin{aligned}
B_{t+2} & =Y_{t+1}-C_{t+1}+(1+r) B_{t+1} \\
& =(1+g) Y_{t}-c(1+g) Y_{t}+(1+r)(1+g) B_{t} \\
& =(1+g)\left(Y_{t}-C_{t}+(1+r) B_{t}\right) \\
& =(1+g) B_{t+1}
\end{aligned}
$$

and so on. Hence $B_{s}$ grows at the same rate as income $Y_{s}$, keeping $B_{s} / Y_{s}$ constant. Consumption in period $s$ is given by

$$
\begin{aligned}
C_{s} & =c Y_{s}=(r-g) \frac{B_{t}}{Y_{t}} Y_{s}+Y_{s} \\
& =(r-g) B_{t}(1+g)^{s-t}+Y_{s} \\
& =(r-g) B_{s}+Y_{s}
\end{aligned}
$$

Hence consumption in period $s$ equals current income plus the growthadjusted interest rate on current asset.
4. Now assume that output is a function of the capital stock, $Y_{t}=A_{t} F\left(K_{t}\right)$. The utility function is specified as $U_{t}=\sum_{s=t}^{\infty} \beta^{s-t} u\left(C_{s}\right)$. Use the period $s$ budget constraint to insert for $C_{s}$ in the utility function.
(a) Find the first-order condition with respect to $K_{t+1}$ and $B_{t+1}$
solution The period-by-period budget constraint is now

$$
C_{s}+B_{s+1}+K_{s+1}=A_{s} F\left(K_{s}\right)+K_{s}+(1+r) B_{s}
$$

substitute this into the objective function to get the maximization problem
$\max _{\left\{K_{s+1}, B_{s+1}\right\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} u\left(A_{s} F\left(K_{s}\right)+K_{s}+(1+r) B_{s}-B_{s+1}-K_{s+1}\right)$
The foc wrt $B_{t+1}$ is

$$
u^{\prime}\left(C_{t}\right)=\beta(1+r) u^{\prime}\left(C_{t+1}\right)
$$

and wrt $K_{t+1}$ is

$$
u^{\prime}\left(C_{t}\right)=\beta\left(1+A_{s} F^{\prime}\left(K_{t+1}\right)\right) u^{\prime}\left(C_{t+1}\right)
$$

combining them gives

$$
A_{s} F^{\prime}\left(K_{t+1}\right)=r
$$

So we have the standard Euler equation for consumption + the optimal investment condition.
(b) Suppose productivity is constant $A_{s}=A_{t}$ for all $s \geq t$ and that, by coincidence, $\beta(1+r)=1$. Describe the time-profiles of consumption, investment and the current account (you can assume that initial net foreign assets, $B_{t}$, are zero).
solution Consumption is constant in all periods and investment and current account is equal to zero in all periods but the first. Since $\beta(1+r)=1$ consumption is constant. Let $K^{*}$ be the capital stock that equalizes the marginal product of capital with world market interest rate $r$

$$
A F^{\prime}\left(K^{*}\right)=r
$$

investment in the initial period $t$ is $I_{t}=K_{t+1}-K_{t}=K^{*}-K_{t}$, the sign depends on the initial level of capital. Investment in all future periods is zero. What about the current account $C A_{s}=B_{s+1}-B_{s}$ ? In the initial period $t$ it is given by

$$
C A_{t}=B_{t+1}=Y_{t}-C^{*}-I_{t}
$$

and in all future periods $s>t$ it is given by

$$
C A_{s}=B_{s+1}-B_{s}=Y^{*}-C^{*}+r B_{s}
$$

The only possible value is $C A_{s}=0$ and thus $B_{s}=B_{t+1}$ for all $s \geq t+1$ Why? If $C A_{s}$ is not zero the country will accumulate debt or assets for ever, which either violates the no-Ponzi scheme condition (in the case of debt) or is suboptimal (in the case of assets), thus violating the condition $\lim _{T \rightarrow \infty} \frac{B_{t+T+1}}{(1+r)^{T}}=0$. To see this look at the growth rate of $B_{s+1}$ from the current account. If $Y^{*}-C^{*}>r B_{s}$

$$
\frac{B_{s+1}-B_{s}}{B_{s}}=r+\frac{Y^{*}-C^{*}}{B_{s}}
$$

When $B_{s}$ grows for ever, the term $\frac{Y^{*}-C^{*}}{B_{s}}$ goes to zero, and in the limit $B_{s}$ grows at rate $r$. The initial current account $C A_{t}$ depends on the initial capital stock. If $K_{t}=K^{*}$ then also $C A_{t}=0$. If the country has $K_{t}<K^{*}$, then it will run a initial deficit, and if $K_{t}>K^{*}$ it runs a surplus (same argument as in question 7 in problem set 1)
(c) Sketch the effects on consumption, investment and the current account from
i. An unexpected temporary increase in productivity in period $t+1$ (that only lasts one period)
solution Variables will jump in period $\mathrm{t}+1$ (nothing happens in period $t$ since the change is unexpected). When productivity increases only in period $t+1$, investment does not react (since marginal productivity of capital is back at its normal level the next period). The country thus gets a one-time increase in output. Consumption still constant, but jumps to a higher level.

The jump is less than the one-time increase in output because of consumption smoothing. The country saves a fraction of the temporary high output in international financial markets, hence the current account is positive in period $t+1$ and then zero.
ii. A temporary increase in productivity in $t+1$ (that only lasts one period) that becomes known at the beginning of period $t$
solution Now investment reacts. The country invests in capital in period $t$ to take advantage of high productivity in the next period. Consumption increases immediately to a permanent higher level. To finance both more consumption and investment, the country borrows from abroad by running a current account deficit in period $t$. In period $t+1$ investment is negative since the capital stock returns to $K^{*}$ in period $t+2$ and zero thereafter. The current account in period $t+1$ is positive and zero thereafter.
iii. An unexpected permanent increase in productivity
solution Investment adjust immediately in period $t$ and then returns to to zero. Consumption increases permanently. Output increases immediately due to higher productivity. In period $t+1$ in increases even further since the capital stock is now higher. Consumption therefore increases more than output in period $t$ so savings goes down. Both higher investment and lower savings gives a current account deficit in period $t$ and zero thereafter.

