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Hand-out ECON 4335 Economics of Banking

The derivation of the main result in Tobin's model

The liability side: Deposits (D) + Equity (E)

E is assumed exogenous, while D is a random short-term deposit. Uncertainty can be rationalized by banks not rolling over their loans to our bank, depositors might take the money out randomly – you never know when a depositor needs money. D + E is the amount of money that can be lent, extended or provided as loans. We rule out any interest on deposits – this is just a simplifying assumption.

The asset side: Loans (L) + Required Reserves (kD) + Defensive position (R)

It is assumed that the fraction of deposits kept as required reserves, *k*, is a fixed constant, and is a government instrument. Loans or the bank's lending policy is a choice variable chosen so as to maximize expected profits, along with defensive position or free reserves.

The loans are less liquid and of longer maturity than deposits; so the bank might get into liquidity problems if unanticipated withdrawals by depositors or by other banks. Reserves beyond those required, called defensive positions, are kept for precautionary reasons, and are highly liquid (cash, deposits in CB, short-term loans to other banks or government bonds). These positions can be negative if the bank has to borrow in the interbank market or in CB; normally at a higher funding rate than what the bank can achieve in the market as a lender, with a positive defensive position.

Before turning to the main problem, random supply of deposits, let us, as a benchmark, show the behaviour of the bank, with a fixed volume of deposits equal to D_0.

The expected revenue from lending is given, for simplicity, by the strictly increasing and strictly concave revenue function $P(L)$, with $P(0)=0, P'(L)>0, P''(L)<0$; we also assume $P'(0)=\infty$ and $P'(\infty)=0$ (to guarantee interior solution). This function can be regarded as a reduced form of the product of the revenue if non-default by the borrowers and the probability for this event.

Profits per year: $\Pi = P(L)+Y(R)$, where we have to take into account the balance sheet condition: $D_0 + E = kD_0 + R + L \Leftrightarrow R = (1-k)D_0 + E - L$. The funding cost of being in a negative defensive position is higher than the interest on revenue from a positive defensive position, but we ignore fixed costs – this is discussed by Tobin. We have:

$$Y(R) = \begin{cases} rR & for\ R \geq 0 \\ (r+b)R & for\ R < 0\ and\ b > 0 \end{cases}$$

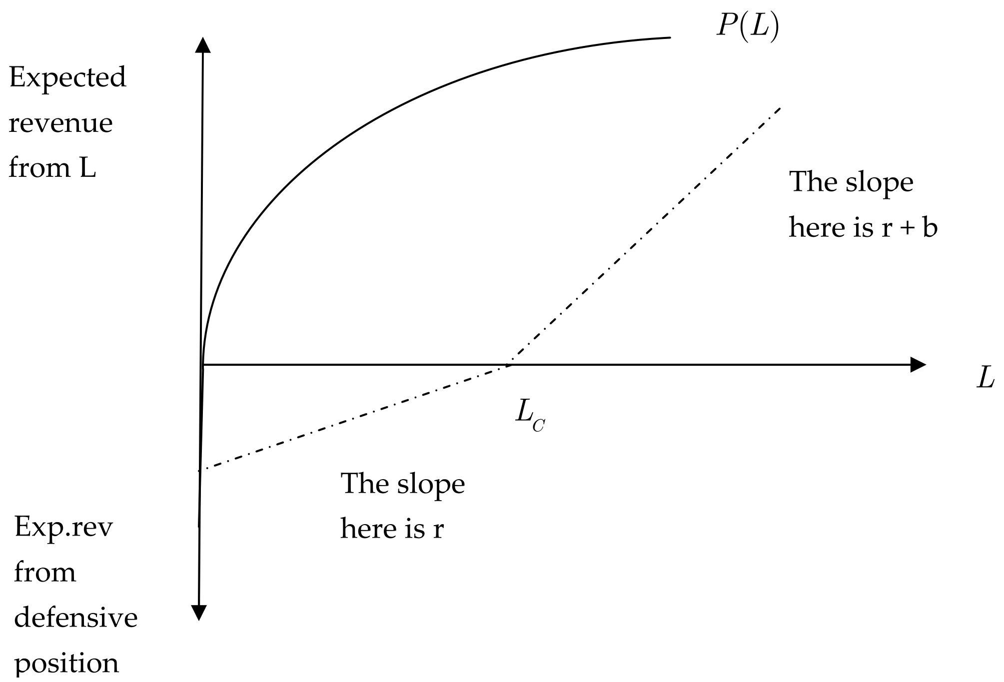
Hence we can write the profit function as consisting of two parts; one for each "regime", depending on the sign its defensive position:

If positive defensive position: $\Pi_+(L) = P(L) + r \cdot \left[(1-k)D_0 + E - L\right]$

If negative defensive position: $\Pi\_(L) = P(L) + (r+b) \cdot \left[(1-k)D_0 + E - L\right]$

The opportunity cost of making loans is r if $R \geq 0$, and $r+b$ if $R < 0$.

Define a critical volume of loans as given by L_C correponding to defensive positions exactly equal to zero; $L_C = (1-k)D_0 + E$. At this loan volume, the derivative of the opportunity cost curve is discontinuous. We can illustrate this in a diagram:



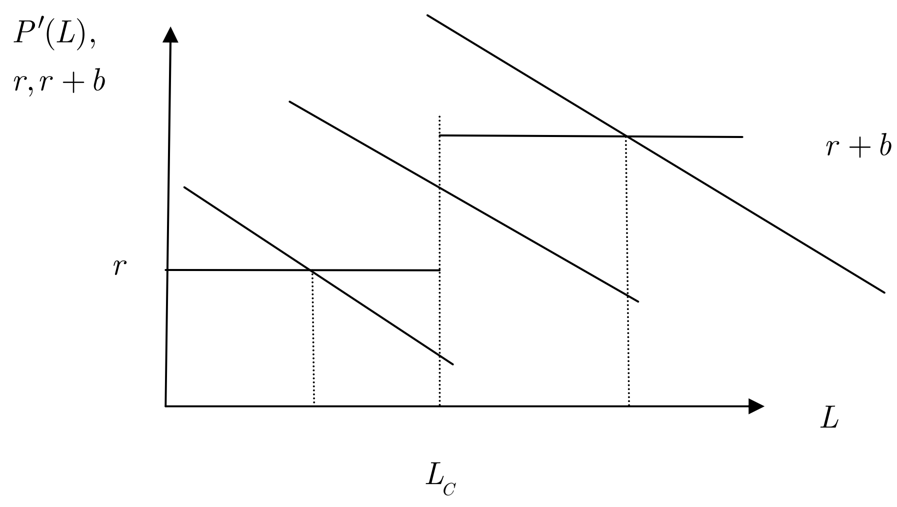
Depending on whether $P'(L_C)$ is below or above r or above $r+b$, the optimal amount of loans, L^*, is determined from the optimality condition:

If $P'(L_C) \leq r$, then $P'(L^*) = r$ for some $L^* \leq L_C$, and $R^* \geq 0$, because $P(L)$ is concave.

If $P'(L_C) \in (r, r+b)$, then $L^* = L_C$, and $R^* = 0$.

At last, if $P'(L_C) \geq r+b$, then $L^* \geq L_C$ and $R^* \leq 0$

These alternatives are illustrated in the diagram below, with three different levels of the marginal revenue from lending:



Let us turn to the interesting case: *Supply of deposits being random*. In this case we expect the bank to hold excessive reserves as a buffer against unexpected withdrawals from depositors. Hence, the bank has to take into account that withdrawals can be so high that even with small amount of loans the defensive position might be negative with some probability. In the diagram above, the probability for a negative position for loans less than L_C, was zero. When deposit withdrawals are uncertain, the probability of ending up in a negative position might be positive even for a cautious or conservative lending policy. We will show that when we introduce a probability distribution for withdrawals, the discontinuous marginal opportunity cost curve in the certainty case is replaced by a continuous one, showing the expected opportunity cost of making loans. This expected opportunity cost will for any L include an allowance for having a negative defensive position with some probability; this allowance will be higher the higher is L.

The bank is risk neutral, maximizing expected net revenue. Let deposits be random, given by the stochastic variable $D = D_0(1+X)$, where $X \geq -1$ and $E(X) = 0$; hence $E(D) = D_0$ is a constant. The cumulative probability distribution for X is

$F(x) = \Pr(X \leq x)$; differentiable and strictly increasing; hence positive density $f(x) = \frac{dF(x)}{dx}$. L is set *before* deposits are known; hence defensive position $R = (1-k)D_0(1+X) + E - L$ is also a random variable.

Define a critical level so that $R \leq 0$. (I use a somewhat different notation than what Tobin does.)

We have $R \leq 0$ for $XD_0 \leq yD_0 = \frac{L-E}{1-k} - D_0 \Rightarrow y = -1 + \frac{L-E}{(1-k)D_0}$. Expected reserves are $E(R) = (1-k)D_0 + E - L = (1-k)\left[D_0 - \frac{L-E}{1-k}\right] = -yD_0(1-k)$. Note that $yD_0 = -\frac{E(R)}{1-k}$, and $E(X) = \int_{-1}^{\infty} xf(x)dx = 0$. Hence we have:

$R \leq 0$ if $X \leq y$, with the higher opportunity cost, and $R > 0$ for $X > y$, with the lower opportunity cost. Therefore $\Pr(R \leq 0) = \Pr(X \leq y) = F(y) = F(\frac{L-E}{(1-k)D_0} - 1)$

Define expected profits per year as:

$$\Pi(L,y) = \int_{-1}^{y} \left\{P(L) + (r+b)\left[(1+x)D_0(1-k) + E - L\right]\right\} f(x)dx$$
$$+ \int_{y}^{\infty} \left\{P(L) + r\left[(1+xD_0(1-k) + E - L\right]\right\} f(x)dx$$

which can be written as (I take it slowly, step by step):

$$\Pi(L,y) = P(L) + \int_{-1}^{y} (r+b)(1-k)\left[\overbrace{\frac{E-L}{1-k} + D_0}^{-yD_0} + xD_0\right] f(x)dx$$
$$+\int_{y}^{\infty} r(1-k)\left[\frac{E-L}{1-k} + D_0 + xD_0\right] f(x)dx$$

$$= P(L) + \int_{-1}^{y} (r+b)(1-k)D_0\left[x-y\right] f(x)dx + \int_{y}^{\infty} r(1-k)D_0\left[x-y\right] f(x)dx$$

$$= P(L) - (r+b)(1-k)D_0 y\int_{-1}^{y} f(x)dx + (r+b)(1-k)D_0\int_{-1}^{y} xf(x)dx$$

$$-r(1-k)D_0 y\int_{y}^{\infty} f(x)dx + r(1-k)D_0\int_{y}^{\infty} xf(x)dx$$

$$= P(L) - (r+b)(1-k)D_0 yF(y) + r(1-k)D_0\int_{-1}^{y} xf(x)dx + b(1-k)D_0\int_{-1}^{y} xf(x)dx$$

$$-r(1-k)D_0 y(1-F(y)) + r(1-k)D_0\int_{y}^{\infty} xf(x)dx$$

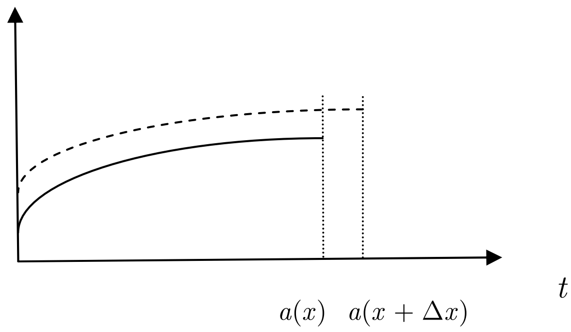
$$= P(L) - r(1-k)D_0 yF(y) - b(1-k)D_0 yF(y)$$
$$+r(1-k)D_0\underbrace{\int_{-1}^{\infty} xf(x)dx}_{E(X)=0} - r(1-k)yD_0(1-F(y)) + b(1-k)D_0\int_{-1}^{y} xf(x)dx$$

Hence we get at last:

$\Pi(L,y) = P(L) - r(1-k)yD_0 - b(1-k)D_0 yF(y) + b(1-k)D_0\int_{-1}^{y} xf(x)dx$, where we have:

$$y := -1 + \frac{L-E}{(1-k)D_0} \Rightarrow \frac{\partial y}{\partial L} = \frac{1}{(1-k)D_0} > 0 .$$

To maximize $\Pi(L, y)$ when y itself is a function of L and enters as the upper limit in the integral (the last term of the objective function), we need a little detour to the following problem: Let $G(x) = \int_0^{a(x)} g(t, x)dt$, which can be illustrated as the area below the graph for the g-function and the horiozontal axis between the origo and the end-point $a(x)$ as below:



Suppose now (this is a bit more general than the problem we are solving) that if x increases, both $a(x)$ and $g(t, x)$ will increase; the upper limit is pushed towards the right along the horizontal axis, from $a(x)$ to $a(x + \Delta x)$ whereas the graph itself will shift upwards for any t; hence we assume (given differentiability) that $\frac{\partial g(t, x)}{\partial x} := g_x(t, x) > 0$. (In our problem we have $g_x(t, x) = 0$.) The new curve is the one shown by the dotted graph. For a finite increase in x, the increase in the function G is the area between the two graphs for a fixed end-point, plus the area as shown by the rectangle between the old and the new end-point. This is the "explanation" for the Leibniz formula as given by $G'(x) = g(a(x), x) \cdot a'(x) + \int_0^{a(x)} g_x(t, x)dt$, where the first term "is the rectangle", whereas the last term is the area between the graphs for a given end-point as given by the original one. In our problem, the last term is zero;

hence we have $G'(x) = g(a(x), x) \cdot a'(x)$, which is a rectangle with height equal to the value of g at $a(x)$; i.e. $g(a(x), x)$, multiplied by the increase in the distance between $a(x)$ and $a(x + \Delta x)$ along the horizontal axis as $\Delta x \to 0$. Hence the derivative $G'(x)$ is the increase as shown by a very "thin" rectangle with height $g(a(x), x)$ and very small width $a'(x)\Delta x$; as x increases by Δx, the end-point itself will increase by $a'(x)\Delta x$.

Let us therefore return to Tobin. The objective is to choose L so as to maximize expected profits, with a FOC, derived from using Leibniz' formula on the last term, when we also use that $dF = f'(x)dx$ or $F'(x) = f(x)$:

$$\frac{\partial \Pi}{\partial L} = P'(L) - r(1-k)D_0 \frac{\partial y}{\partial L} - b(1-k)D_0 \left[F(y)\frac{\partial y}{\partial L} + yf(y)\frac{\partial y}{\partial L} \right] + b(1-k)D_0 yf(y)\frac{\partial y}{\partial L} = 0$$

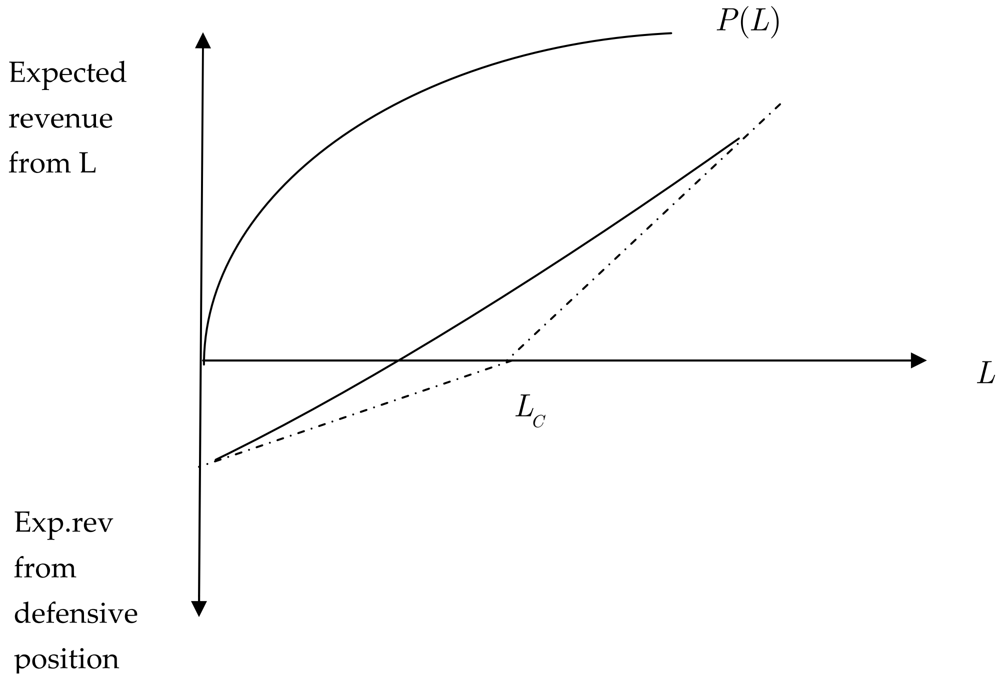
Use that $(1-k)D_0 \dfrac{\partial y}{\partial L} = 1$ from above and collect terms, we get our result reported on Monday in class: The volume of loans that maximizes expected profits, $\tilde{L}$, is determined from the FOC:

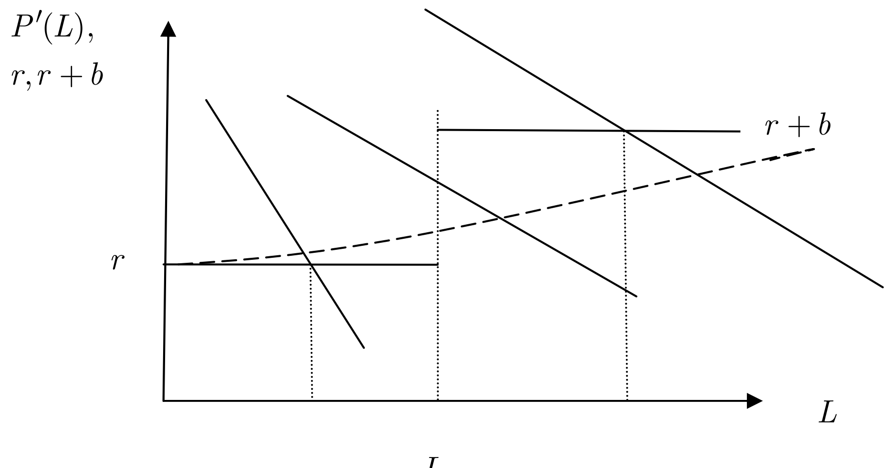
$P'(\tilde{L}) - r - bF(y) = 0$ or

$P'(\tilde{L}) = r + bF(y)$; where the RHS is the expected opportunity cost of lending.

Expected marginal revenue from making loans should be equal to marginal expected opportunity cost of lending. Because F is increasing, this additional term is higher the higher is L, as y itself is increasing in L: The marginal opportunity cost is therefore increasing in L.

This means that this expected marginal cost of lending, if f goes to zero as x goes to infinity, can be illustrated as:





Here the dashed curve is $r + bF(y)$; starting at r an approaching r+b.