4335 S1 H14 Tobin's Model

1. Opt. conditions with random deposits

$D+E = kD + R + L$ (1) Accounting identity

Return from lending : $p(L)$

Return from defensive position : $Y(R) = \begin{cases} rR & R \geq 0 \\ (r+b)R & R<0 \end{cases}$

$D = (1+X)D_0$, $E(X)=0$ s.t. $E(D) = D_0$. $X \geq -1$, $D_0>0$ s.t. $D \geq 0$

$X \sim F(x) = P(X \leq x)$, $f(x) \equiv \frac{dF(x)}{dx} > 0$ pdf of X.

Before optimization, we need to know for which realizations of X is R negative s.t. we know its return fn.

$R \underset{(1)}{=} (1-k)D+E-L = (1-k)(1+X)D_0 + E - L = 0$

(2) $x_c = \frac{L-E}{(1-k)D_0} - 1$ is the critical value for which $R=0$

For $X \geq x_c$, $Y(R) = rR$; $X < x_c$, $Y(R) = (r+b)R$

$E(R) = (1-k)D_0 + E - L = -(1-k)D_0 x_c$ is the expected defensive position.

The bank's profit is returns from loans and defensive positions added together:

$$\begin{aligned}
\pi(L, x_c(L)) &= p(L) + Y(R) \\
&= p(L) + \int_{-1}^{x_c} (r+b)R(x)f(x)\,dx + \int_{x_c}^{\infty} rR(x)f(x)\,dx \\
&= p(L) + (r+b)\int_{-1}^{x_c} [(1-k)(1+X)D_0 + E - L] f(x)\,dx + r\int_{x_c}^{\infty} [(1-k)(1+X)D_0 + E - L] f(x)\,dx \\
&= p(L) + E(R)[(r+b)F(x_c) + r(1-F(x_c))] + (1-k)D_0[(r+b)\int_{-1}^{x_c} xf(x)\,dx + r\int_{x_c}^{\infty} xf(x)\,dx] \\
&= p(L) - (1-k)D_0 x_c [(r+b)F(x_c) + r(1-F(x_c))] + (1-k)D_0 [r\int_{-1}^{\infty} xf(x)\,dx + b\int_{-1}^{x_c} xf(x)\,dx] \\
&= p(L) + (1-k)D_0\{-x_c[(r+b)F(x_c) + r(1-F(x_c))] + (r \cdot \underset{0}{E(X)} + b\int_{-1}^{x_c} xf(x)\,dx)\} \\
&= p(L) + (1-k)D_0\{b\int_{-1}^{x_c} xf(x)\,dx - x_c[bF(x_c) + r]\} \qquad (3)
\end{aligned}$$

Since the distribution of X is known, x_c is just a number, $F(x_c)$ a probability (of $R<0$), Expected profit is a fn. of loans L and x_c, which is a fn. of L too and $\frac{\partial x_c}{\partial L} = \frac{1}{(1-k)D_0}$ from (2).

Under certainty, $x=0$ s.t. $\pi(L, D(0)) = \pi(L)$ Optimal condition:

$$\pi(L) = p(L) \underbrace{- (1-k)D_0 x_c}_{=R = A^l - L} \cdot \begin{cases} r, & R \geq 0 \\ r+b, & R<0 \end{cases} \quad \begin{array}{l} , \ L^* \text{ defined as } p'(L^*) = r \\ , \ L^* \text{ defined as } p'(L^*) = r+b \end{array} \qquad (4)$$

Liquid asset $A^l = (1-k)D_0 + E = A - kD_0$

Under uncertainty, maximization involves the use of Leibniz's formula

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\,dt = f(x, v(x))\,v'(x) - f(x, u(x))\,u'(x) + \int_{u(x)}^{v(x)} f'_x(x,t)\,dt.$$

$$\frac{d}{dL}\int_{-1}^{x_c(L)} xf(x)\,dx = x_c f(x_c)\frac{\partial x_c}{\partial L}$$ since -1 is a number and x is independent of L.

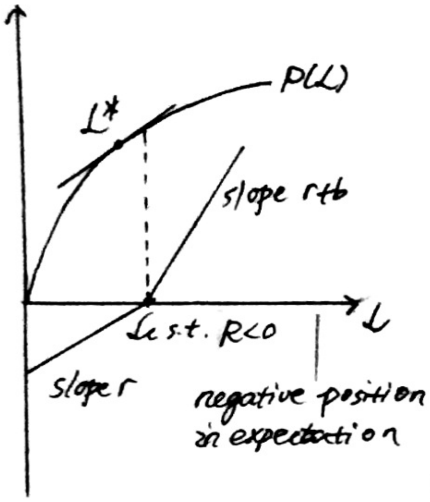
$$\begin{aligned}
\pi'(L) &= p'(L) + (1-k)D_0\left[\frac{bx_c f(x_c)}{(1-k)D_0} - x_c bF'(x_c)\frac{1}{(1-k)D_0} - \frac{1}{(1-k)D_0}(bF(x_c)+r)\right] \\
&= p'(L) + bx_c f(x_c) - x_c b f(x_c) - [bF(x_c) + r] \\
&= p'(L) - [bF(x_c) + r] = 0
\end{aligned}$$

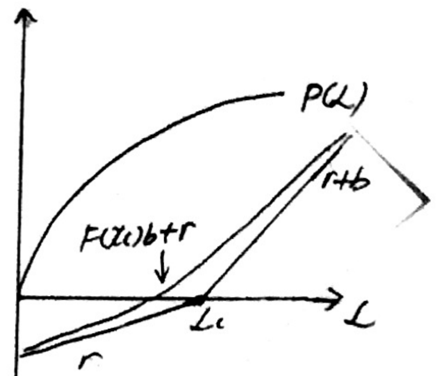
FOC: $p'(L^*) = bF(x_c) + r$ where $F(x_c(L))$ is $P(R<0)$. (5)

Comparing (4) and (5): they are very similar, only the opt. L^* in (4) is discontinuous.

It essentially involves equating marginal returns on L and R.

Graphs : certainty uncertainty





In this graph optimal asset allocation is $R>0$. With a lower r $p'(L)=r$ will be on the right side of L_c, which isn't allowed, and we will have to set $p'(L^*)=r+b$ and have $R<0$. If $L^*=L_c$, we have an unstable equilibrium

$F(x_c)b+r$ approaches r as $L\to 0$, $r+b$ as $L\to\infty$. The new return curve on R, the defensive asset, is a smoothed version of the previous one. Even if $L<L_c$, the defensive position may be negative due to a shock to X.

2. Comparative Statics

Certainty : $p'(L^*)=r$, $R\geq 0$
$p'(L^*)=r+b$, $R<0$

uncertainty: $p'(L^*) = bF(x_c)+r$

* Higher D_0 : higher L_c, the return on R curve shifts to the right, more lending at optimum.
* Higher E : same qualitative effect, but bigger as E is 100% liquid asset.
* Lower $p(L)^e$: a flatter $P(L)$ curve. More likely to stay on the safe side ($L^*<L_c$), Lower L^*.
* Liquidity dry-up : b increases. More likely to have $R>0$. Lend less. L^* smaller.

3. Value of Equity

Find value fn. by inserting opt. conditions back into π.

Certainty :

$$\pi(L^*,E)=\begin{cases} p(L^*)-[\overbrace{L^*-E-(1-k)D_0}^{R}]\cdot r & R\geq 0 \\ p(L^*)-[L^*-E-(1-k)D_0]\cdot(r+b) & R<0 \end{cases} \qquad \frac{\partial\pi(L^*,E)}{\partial E}=r, \quad \frac{\partial\pi(L^*,E)}{\partial E}=r+b \qquad (7)$$

Uncertainty:

$$\pi(L^*,E)= p(L^*)+(1-k)D_0\left[b\int_{-1}^{x_c(L^*,E)} x f(x)\,dx - x_c(L^*,E)\left(bF(x_c(L^*,E))+r\right)\right]$$

$$\frac{\partial\pi(L^*,E)}{\partial E} = p'(L^*)\cdot bf(x_c)\cdot\frac{\partial x_c(L^*,E)}{\partial E} + (1-k)D_0\left\{\frac{bx_cf(x_c)}{(1-k)D_0} - \frac{1}{(1-k)D_0}(bF(x_c)+r) - \frac{x_cbf(x_c)}{(1-k)D_0}\right\}$$

Here L^* depends on E directly

$$=\frac{[bF(x_c)+r]\,bf(x_c)}{(1-k)D_0} + bx_cf(x_c) - bx_cf(x_c) - \frac{[bF(x_c+r)]}{(1-k)D_0}$$

$$=\frac{bF(x_c)+r}{(1-k)D_0}\cdot(bf(x_c)-1) \qquad (8)$$

4. Riskier X, e.g. higher σ makes $F(x)b+r$ curve flatter s.t. banks will be more cautious and lend less.