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Lecture notes ECON 4350

The "neoclassical" version of the Johansen-vintage (putty-clay) growth model

The various elements we have considered during the lectures can be collected so as to give us the following set of equations, based on "neoclassical" assumptions - an exogenous saving rate (connecting current income to the purchase of *new* equipment at t) and exponential growth in total labour force. (The present model is more in line with the version proposed by Phelps.) We also assume one commodity being produced in this economy. This good is used to produce new capital equipment; and units of measurement have been chosen so that one unit of the final good is used to produce one unit of equipment. The price of the final good as well as capital equipment is set equal to one. Then we have the following model, when we impose an anticipated real wage schedule, $\dot{w}(\tau) = g \cdot w(\tau)$, where g is interpreted as the expected rate of change:

(1) $$X(t) = \int_{t-\theta}^{t} x(\tau) d\tau$$

(2) $$N(t) = \int_{t-\theta}^{t} n(\tau) d\tau$$

(3) $$N(t) = N_0 e^{\nu t}$$

(4) $$x(t) = f(n(t), k(t), t)$$

(5) $$k(t) = sX(t)$$

(6) $$x(t-\theta) = w(t) \cdot n(t-\theta)$$

(7) $$f_k \int_{t}^{t+\theta} e^{-r(\tau - t)} d\tau - 1 = 0 \Leftrightarrow f_k = \frac{r}{1 - e^{-r\theta}} \approx r + \frac{1}{\theta}$$

(8) $$f_n \int_{t}^{t+\theta} e^{-r(\tau - t)} d\tau - \int_{t}^{t+\theta} e^{-r(\tau - t)} w(t) e^{g(\tau - t)} d\tau \Leftrightarrow f_n = \frac{r}{r-g} \frac{1 - e^{-(r-g)\theta}}{1 - e^{-r\theta}} \cdot w(t)$$

Equation (1) is a supply function for total output of the final good, available at t (equal to income), and is made up of output from all plants of different vintages of equipment being used at t.[1] Equation (2) defines total labour requirement on all

[1] When we talk about a vintage, we have to be a bit careful when time is continuous. Equipment of vintage τ is the amount of installed equipment during the short interval [$\tau, \tau + d\tau$], $k(\tau)d\tau,$ which is combined with the number of man-hours on this equipment, as given by $n(\tau)d\tau$, to produce an

plants being operated at t. Relation (3) shows demographic dynamics; population or labour force increases at an exogenous rate equal to $100\nu\%$ per unit of time at t. (2) and (3) constitute an equilibrium condition in the labour market, where demand (2) is equal to supply of labour at t as given by (3).

In (4) we have the ex ante function or all available techniques or modes of production at t ("choice-of-technique function" at t), taking into account that technological progress is *embodied*; as given by the index "t" in the production function. New techniques of production or know-how can only be utilized by investing in new equipment produced at t. (We neglect the fact that physical capital equipment normally is completed during some production period.) In (5) we have that saving at t is a fixed fraction of total income at t, and this is automatically invested in *new equipment* at t. (This condition is an equilibrium condition, as required saving equals investment.) In (6) we have the scrapping condition, defining the oldest plant in use at t. The age of the oldest operating plant at t is θ, i.e. of the vintage, $t-\theta$, that earns zero quasi-rent. At last (7) and (8) are first-order conditions for an entrepreneur maximising the difference between the present discounted value of expected future quasi-rents and the cost of purchasing new equipment at any t, over the expected life-span of the plant, taking into account that the production technique and capacity will be fixed as long as the plant is operated. The objective at *t* is:

$$Max_{(n,k)} \int_{t}^{t+\theta} e^{-r(\tau-t)} \Big[f(n(t),k(t),t) - w(\tau)n(t) \Big] d\tau - k(t)$$

when assuming that the entrepreneur can perfectly predict the "true" life-span of the plant. We can then consider (7) as demand for new equipment at t, whereas (8) is the demand for labour to be used along with equipment installed at t.

The model has 8 equations and 8 variables, determining $X, N, x, n, k, w, r, \theta$ for any t; hence it seems to have a solution. The prices *w* and *r* will be equilibrium prices, taking such values so that we have equilibrium in the labour market and that demand for new equipment is compatible with the saving (the fraction of the total output that is invested). In this model we should have to take into account that $\theta = \theta(t)$ and also that we have to incorporate how expectations are formed.[2] The wage path that the entrepreneur uses in his calculations will be based upon what he or she anticipates about the future. In fact the expected growth rate *g* in the wage function is exogenous, and the solution will therefore depend on what the

output equal to $x(\tau)d\tau$. Note also that θ is an endogenous variable and hence should be written as a function of t.

[2] The role of price expectations in a similar type of model, however with declining capital efficiency over time, is analysed both theoretically and numerically, in E. Biørn and P. Frenger, 1992, "Expectations, Substitution, and Scrapping in a Putty-Clay Model"; Journal of Economics (Zeitschrift für Nationalökonomie) 56 (2), pp. 157-184.

entrepreneurs anticipate about future prices and future rate of interest, as well.[3] (Here we assume that *r* is independent of t.) There is no explicit depreciation or decay of capital equipment as long as the plant or equipment still earns a positive quasi-rent. Hence as long as the equipment is operating, it is intact in the same way as it was when it was installed. (We have "sudden death" due to obsolescence, and not due to the "sudden death" coming from a fixed technical lifetime of the equipment.) There is no disembodied technical progress that can benefit a producer who is stuck with his or her past choice.

In this model, with embodied technical progress, the role of saving is important in the sense that investment will turn up as new and modern equipment, which will replace older, less effective equipment. When older equipment is scrapped, relatively more labour will be released per unit equipment being scrapped. (Note that at the plant level, with real wage expected to increase, marginal productivity of labour hired and combined with new equipment at t, will exceed the current wage at t, and therefore also exceed the average productivity of labour used on equipment being scrapped at t.)

We now ask the following questions:

- Will the model exhibit a "balanced equilibrium character" in the sense of a partially balanced growth path, as both total output and "vintage" output grow at the same rate, so as to keep (1) satisfied for all t? Do we have a similar requirement as to labour force and total labour requirement for vintages in use; i.e. so as to have $N_0 e^{\nu t} = \int_{t-\theta}^{t} n(\tau) d\tau$ for any t?
- If "yes", what is the implied growth rate for the output?

The first question will be answered when we assume that the ex ante function is of the Cobb-Douglas type; $x(t) = f(n(t), k(t), t) = An(t)^a k(t)^b e^{\varepsilon t}$ with $a + b = 1$, where the term $e^{\varepsilon t}$ shows the efficiency of the capital equipment, along with the applied labour, at the date when new equipment is installed. (This term captures the level of know-how at the date when new equipment is installed; this rate of embodied technological growth is exogenous and given by $100\varepsilon\%$ per unit of time.)

The working hypothesis is the: Can we find a growth rate *G* so that (1) is satisfied for all t; i.e., with $X(t) = X(0)e^{Gt}$ and $x(t) = x(0)e^{Gt}$?

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[3] An early contribution discussing the role of expectations in the vintage model is Kemp and Thãn, "On a class of Growth Models", *Econometrica,* Vol. 34, No. 2 (April; 1966), pp. 257-282. See also E.Phelps "Substitution, Fixed Proportions, Growth and Distribution", *International Economic Review*, Vol. 4 (Sept., 1963), pp. 265-288.

If this hypothesis is correct, and when we suppose θ to be fixed, we have:

$$X(0)e^{Gt} = x(0)\int_{t-\theta}^{t} e^{G\tau}d\tau = x(0)\left[\frac{1}{G}e^{G\tau}\right]_{t-\theta}^{t} = \frac{x(0)}{G}\left[e^{Gt} - e^{G(t-\theta)}\right] \Leftrightarrow X(0) = \frac{x(0)}{G}\left[1 - e^{-G\theta}\right]$$

If the initial values are related to each other so that $X(0) = \frac{x(0)}{G}\left[1 - e^{-G\theta}\right]$, then we have balanced growth in new capacity and in aggregate output, so as to have (1) satisfied for all t.

We furthermore have: $N(t) = N_0 e^{\nu t}$. For the required labour to be used along with new equipment installed at t to grow at the same rate as total labour force; $n(t) = n_0 e^{\nu t}$, the following must hold, when θ is fixed:

$$N(t) = N_0 e^{\nu t} = \int_{t-\theta}^{t} n(\tau)d\tau = n_0 \int_{t-\theta}^{t} e^{\nu\tau}d\tau = \frac{n_0}{\nu}\left[e^{\nu\tau}\right]_{t-\theta}^{t} = \frac{n_0}{\nu}e^{\nu t}\left[1 - e^{-\nu\theta}\right]$$

Hence; if $N_0 = \frac{n_0}{\nu}\left[1 - e^{-\nu\theta}\right]$, then both *N* and *n* will exhibit balanced growth as well, and grow at a rate $100\nu\%$ per unit of time.

From (5) we then have $k(t) = sX(t) = sX(0)e^{Gt} = k_0 e^{Gt}$, telling us that new equipment will grow at the rate $100G\%$ per unit of time, as well.

Using these conditions in the "increment" function, we get directly a solution for the growth rate *G*:

$$x(t) = An(t)^a k(t)^b e^{\varepsilon t} \Rightarrow x(0)e^{Gt} = A\left[n_0 e^{\nu t}\right]^a \left[k_0 e^{Gt}\right]^b e^{\varepsilon t} = An_0^a k_0^b e^{(a\nu + bG + \varepsilon)t}$$
$$= x(0)e^{(a\nu + bG + \varepsilon)t} \Leftrightarrow e^{Gt} = e^{(a\nu + bG + \varepsilon)t} \Leftrightarrow G = a\nu + bG + \varepsilon \Rightarrow G = \frac{a\nu + \varepsilon}{1-b} = \nu + \frac{\varepsilon}{a}$$

The growth rate that will support equilibrium for any t is given by $G = \nu + \frac{\varepsilon}{a}$.

Remark:
If we have the standard neoclassical model with no distinction between ex ante and ex post; where $X(t) = AN(t)^a K(t)^b e^{\varepsilon t}$, $\dot{K}(t) = sX(t)$ *and* $N(t) = N(0)e^{\eta t}$, and $a + b = 1$, we get: $\dot{K}(t) = sA(N(0)e^{\eta t})^a K(t)^b e^{\varepsilon t}$, where the trend term is a disembodied technical progress-component different from the embodied part above; the rate of technical progress needs to be interpreted differently; also the parameters (a,b) must be

interpreted differently! We can write the differential equation as $\dot{K}(t) = \underbrace{sAN_0^a}_{=\Lambda}[K(t)]^b e^{(a\eta+\varepsilon)t}$. This is a Bernoulli differential equation that can be solved using the following formula – you are not expected to solve such a differential equation; consider it as an example: Let

$$Z := K^{1-b} \Rightarrow (1-b)K^{-b}\dot{K} = \dot{Z} \Rightarrow K^{-b}\dot{K} = \frac{1}{1-b}\dot{Z} = \Lambda e^{(a\eta+\varepsilon)t} \Rightarrow \dot{Z}(t) = (1-b)\Lambda e^{(a\eta+\varepsilon)t},$$

Hence we get $Z(t) = \Phi + \frac{sAN_o^a(1-b)}{a\eta+\varepsilon}e^{(a\eta+\varepsilon)t}$ so as to get: $K = Z^{\frac{1}{1-b}}$, or

$$\Rightarrow K(t) = \left[\Phi + \frac{s(1-b)AN_0^a}{a\eta+\varepsilon}e^{(a\eta+\varepsilon)t}\right]^{\frac{1}{1-b}}$$, with

$$K(0) = \left[\Phi + \frac{s(1-b)AN_0^a}{a\eta+\varepsilon}\right]^{\frac{1}{1-b}} \Rightarrow \Phi = K_0^{1-b} - \frac{s(1-b)AN_0^a}{a\eta+\varepsilon}$$. Hence

$$K(t) = \left[K_0^{1-b} + \frac{s(1-b)AN_0^a}{a\eta+\varepsilon}(e^{(a\eta+\varepsilon)t} - 1)\right]^{\frac{1}{1-b}}$$. Inserting this into the macro production function we get: $X(t) = A\left[N_0 e^{\eta t}\right]^a \left[K_0^{1-b} + \frac{s(1-b)AN_0^a}{a\eta+\varepsilon}(e^{(a\eta+\varepsilon)t} - 1)\right]^{\frac{b}{1-b}} e^{\varepsilon t}$.

We can show that $\frac{\dot{X}(t)}{X(t)} \to \frac{a\eta+\varepsilon}{1-b}$. (Equality if $\Phi = 0$, then

$$X(t) = A\left[N_0 e^{\eta t}\right]^a \left[\frac{s(1-b)AN_0^a}{a\eta+\varepsilon}\right]^{\frac{b}{1-b}} e^{[\frac{(a\eta+\varepsilon)b}{1-b}+\varepsilon]t} = AN_0^a \left[\frac{s(1-b)AN_0^a}{a\eta+\varepsilon}\right]^{\frac{b}{1-b}} e^{[a\eta+\frac{(a\eta+\varepsilon)b}{1-b}+\varepsilon]t}$$

$$= AN_0^a \left[\frac{s(1-b)AN_0^a}{a\eta+\varepsilon}\right]^{\frac{b}{1-b}} e^{\frac{a\eta+\varepsilon}{1-b}t}$$)

The *asymptotic* growth rate will be reached earlier in the neoclassical world as compared to the putty-clay world, because in a putty-clay world we start out with capital equipment that cannot be altered momentarily – history prevents us from adopting the most efficient capital equipment at the same time.
(Also the interpretations of the parameters (a,b) should differ; in the neoclassical model these are related to total input use, whereas in the putty-clay model they are related to the increments.)

Let us go back to the putty-clay world. The remaining discussion is a bit tentative, and shows that we can make some inferences:

The marginal plant used at t is determined from the scrapping condition:

$$w(t) = \frac{x(t-\theta)}{n(t-\theta)} = \frac{x(0)e^{G(t-\theta)}}{n_0 e^{\nu(t-\theta)}} = \frac{x(0)}{n_0}e^{(G-\nu)(t-\theta)} = [\frac{x(0)}{n_0}e^{-\theta(G-\nu)}]e^{(G-\nu)t}$$

As θ is constant in equilibrium the term within brackets must be constant, as well. The real wage will then have to increase at $100(G-\nu)\%$ per unit of time. Hence; the real wage will now grow at a rate $\frac{\varepsilon}{1-b}=\frac{\varepsilon}{a}=G-\nu$. This rate of growth should perhaps be anticipated by investors; at least among those entrepreneurs having rational expectations.

What about the age of the oldest plant in use, and what about the equilibrium rate of interest? For any set of parameters, θ is constant, but will change for a set of new parameters. The variables are affected by the expectations the investors have about the wage increase; as given by $\dot{w}(t)=gw(t)$.

Let us collect the remaining equations in our growth model. We have:

$$(7)' \qquad f_k = b\frac{x(t)}{k(t)} = \frac{r}{1-e^{-r\theta}}$$

In this condition we know that both equipment and new productive capacity will grow at the same rate; hence the ratio between x and k is fully determined by the ratio of their initial values; $\frac{x(0)}{k_0}$.

Then we have, on using $a+b=1$: $b\frac{x(0)}{k_0}=b\frac{An_0^a k_0^b}{k_0}=bA\left[\frac{n_0}{k_0}\right]^a=\frac{r}{1-e^{-r\theta}}$ (*)

At last, from (8) we have, when inserting for the equilibrium wage prevailing at t; *w(t)*:

$$(8)' \quad f_n = a\frac{x(t)}{n(t)} = \frac{r}{r-g}\frac{1-e^{-(r-g)\theta}}{1-e^{-r\theta}}w(t) = \frac{r}{r-g}\frac{1-e^{-(r-g)\theta}}{1-e^{-r\theta}}\left[\frac{x(0)}{n_0}e^{-\theta(G-\nu)}\right]e^{(G-\nu)t}$$

When using the scrapping condition and that $G-\nu=\frac{\varepsilon}{1-b}$, we get:

$$a\frac{x(0)}{n_0}e^{(G-\nu)t} = \frac{r}{r-g}\frac{1-e^{-(r-g)\theta}}{1-e^{-r\theta}}\left[\frac{x(0)}{n_0}e^{-\theta(G-\nu)}\right]e^{(G-\nu)t} \Leftrightarrow a = \frac{r}{r-g}\frac{1-e^{-(r-g)\theta}}{1-e^{-r\theta}}e^{-\frac{\varepsilon\theta}{1-b}} \quad (**)$$

The conditions (*) and (**) give us two conditions to determine *r* and θ, both as functions of the expected rate of wage increase *g*.

Consider now two types of expectations formation:

- "Zero foresight" or naïve (static) expectations in the sense that the investors systematically make mistakes; $g = 0$
- "Perfect foresight" or rational expectations in the sense that the investors can "look through" the mechanism of the model and predict correctly the equilibrium rate of increase in real wage; i.e., $g = \frac{\varepsilon}{a}$

If the investors form naïve or static expectations (and never learn), they assume that wage will stay constant, with $g = 0$. Instead of taking into account that labour will be more expensive in the future and then face lower future quasi-rents than expected, the entrepreneurs will at t choose a too labour-intensive technology. When experiencing that real wage in fact is increasing, they will have to scrap the equipment earlier than planned because quasi-rents are exhausted sooner.
From (**) we have, with $g = 0$:

$$a = e^{-\frac{\varepsilon\theta}{1-b}} \Rightarrow \ln a = -\frac{\varepsilon\theta}{1-b}\ln e \Rightarrow \theta = -\frac{a\ln a}{\varepsilon} > 0\text{, as } a < 1 \Leftrightarrow \ln a < 0\,.$$

If the entrepreneurs have naïve expectations, they will choose a too labour-intensive technique for the plant at t. From (**) we then see that the higher is the productivity increase of newly installed equipment, as given by ε, the lower is θ, that is, the lower is the age of the oldest equipment in use – the capital equipment is scrapped more frequently. Here a high ε means a high embodied technical progress that can only be taken advantage of by installing new equipment. (Note that θ does not depend on the saving rate.) If high turnover of equipment due to high growth in technical know-how (through a high rate of embodied technical progress), the shorter duration will installed capital equipment have.

If the investors form rational expectations or have perfect foresight as to the true rate of increase in real wage, they anticipate correctly that $g = \frac{\varepsilon}{a}$. Given this expectations, the entrepreneurs will choose a capital-intensive technique, combined with labour so that the marginal productivity of labour at t is above the current wage rate at t. This makes, cet.par., the expected life-span of the plant installed at t longer. Less labour hired at t will prolong the period over which the firm can earn positive quasi-rents. (The higher is the expected rate of increase in real wage, the more capital-intensive technique will be chosen at t.)

Note also that if $\varepsilon \to 0$, then the correctly anticipated rate of wage increase, $g = \frac{\varepsilon}{a} \to 0$ as well. And as we noted above, for the case when one correctly anticipates that $g = 0$, we have $\theta \to \infty$, as $\varepsilon \to 0$. If there is no embodied technical progress, the rationally expected wage will be constant, motivating the entrepreneur to choose plant with a very long duration and much longer than what is implied by the technical lifetime. Hence, in this case, scrapping is a "technical decision", and the technical lifetime will then set a limit on the plant's lifespan.

In another very interesting article from 1967, on "Some Problems of Pricing and Optimal Choice of Factor Proportions in a Dynamic Setting", *Economica*, Vol. 34 (May), pp. 131-152, Leif Johansen derived a set of efficiency conditions within a similar, but a more disaggregated setting than the macro-version above. One problem in this dynamic setting, with fixed factor proportions ex post, is whether dynamic optimality can be realised in a system with decentralised decisions, when decisions are highly influenced by expectations about future prices. As he said: "For practical purposes one must therefore conclude that it is necessary to have consistent and correct expectations about future prices in order that decentralized decisions with profit maximization shall lead to fulfilment of the optimality requirements." He was not too optimistic as to whether a free market would cope with that issue – he suggested some market-administrator or planning body which should publish predicted prices; hence his strong interest in so-called indicative planning. Leif Johansen was ahead of "modern" dynamic theories that rely heavily on "futures markets" and rational expectations. But as we know today, for these markets to function well, some very strong assumptions have to be imposed, as both uncertainty and private information will obscure the role of transmitting information through the price system.