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Lecture Notes, ECON 4350

**Some remarks related to multi-sectoral growth models**

This short note provides some remarks as to the term we called $G$; the "gains from reallocation or transfer of labour among sectors", and also to Leif Johansen's MSG-model.

1. *The gains from redistribution of labour*

The point of departure is the model from Vislie (1979). The sector we focus on is sector 1; then sector 2 captures the remaining economy. The economy is closed:

(1) $X_i = F(N_i, t) = A_i N_i e^{\gamma_i t}$ for *i=1,2* Production functions

(2) $N_1 + N_2 = N$ Full emplyment

(3) $\dot{N}(t) = nN(t)$ Growth in labour force, proportional to population

(4) $C_i = N \cdot f_i(p_1, p_2, y)$ ; for $i = 1,2$, $y$ is average income and $p_i$ is nominal price

(5) $w_i N_i = p_i X_i$; or $p_i = \dfrac{w_i}{A_i} e^{-\gamma_i t}$ or $\dfrac{\dot{p}_i}{p_i} = \dfrac{\dot{w}_i}{w_i} - \gamma_i := g_i - \gamma_i$

We have due to the economy being closed that $X_i = C_i$. For sector 1, we have:

(6) $A_1 N_1 e^{\gamma_1 t} = N \cdot f_1(p_1, p_2, y)$ where $Ny = w_1 N_1 + w_2 N_2 := Y$ is total income.

(Hidden in the demand functions, which are homogenous of degree zero, is the budget constraint $\sum_j p_j C_j = Y$.)

Assume all variables being differentiable functions of time. Then from (6), we have:

(7) $\dfrac{\dot{N}_1}{N_1} + \gamma_1 = n + e_{11}\dfrac{\dot{p}_1}{p_1} + e_{12}\dfrac{\dot{p}_2}{p_2} + E_1\dfrac{\dot{y}}{y}$

where we have introduced standard price and income elasticities: $e_{ij} = ElC_i : p_j$ and $E_i := ElC_i : y$.

We have by definition and by specification of the model that nominal income is fully distributed as wage income: $w_1 N_1 + w_2 N_2 = Y$; hence

$\dot{Y} = N_1\dot{w}_1 + w_1\dot{N}_1 + N_2\dot{w}_2 + w_2\dot{N}_2 \Rightarrow \frac{\dot{Y}}{Y} = \frac{w_1N_1}{Y}(\frac{\dot{w}_1}{w_1} + \frac{\dot{N}_1}{N_1}) + \frac{w_2N_2}{Y}(\frac{\dot{w}_2}{w_2} + \frac{\dot{N}_2}{N_2})$. Of course we have: $\frac{\dot{y}}{y} = \frac{\dot{Y}}{Y} - \frac{\dot{N}}{N} = \frac{\dot{Y}}{Y} - n$. Introduce the budget shares: $\alpha_j := \frac{p_jC_j}{Y} = \frac{w_jN_j}{Y}$.

Then we can (when all prices and income are measured in units of some unspecified good) find an expression for how employment in sector 1 develops relative to total employment:

$$\frac{\dot{N}_1}{N_1} - n = -\gamma_1 + e_{11}\cdot(g_1 - \gamma_1) + E_1\cdot(\frac{\dot{Y}}{Y} - n) + e_{12}\cdot(g_2 - \gamma_2)$$
$$= -\gamma_1 + e_{11}\cdot(g_1 - \gamma_1) + E_1\cdot\left[\alpha_1\cdot(g_1 + \frac{\dot{N}_1}{N_1}) + \alpha_2\cdot(g_2 + \frac{\dot{N}_2}{N_2})\right] - nE_1 + e_{12}\cdot(g_2 - \gamma_2) \qquad (*)$$

From (2) we get: $\frac{\dot{N}_2}{N_2} = n + (n - \frac{\dot{N}_1}{N_1})\cdot\frac{N_1}{N_2}$, which we insert in (*)

$$\frac{\dot{N}_1}{N_1} - n = -\gamma_1 + e_{11}\cdot(g_1 - \gamma_1) + e_{12}\cdot(g_2 - \gamma_2) - nE_1$$
$$+E_1\cdot\left[\alpha_1\cdot(g_1 + \frac{\dot{N}_1}{N_1}) + \alpha_2\cdot(g_2 + n + (n - \frac{\dot{N}_1}{N_1})\cdot\frac{N_1}{N_2})\right]$$

Then, on rearranging terms we get:

$$\frac{\dot{N}_1}{N_1}\cdot\left[1 - \alpha_1E_1 + \alpha_2E_1\frac{N_1}{N_2}\right] - n$$
$$= -\gamma_1 + e_{11}\cdot(g_1 - \gamma_1) + e_{12}\cdot(g_2 - \gamma_2) - nE_1 + \alpha_1E_1g_1 + \alpha_2E_1g_2 + \alpha_2nE_1(1 - \frac{N_1}{N_2})$$

which when using that $\alpha_2 = 1 - \alpha_1$ and the Slutsky equation: $e_{ij} = S_{ij} - \alpha_jE_i$, with $S_{ij}$ as the Slutsky-elasticity, with the property that $\sum_j S_{ij} = 0$ and $S_{ii} < 0$, can be written as:

$$\frac{\dot N_1}{N_1}\cdot\left[1-\alpha_1E_1+\alpha_2E_1\frac{N_1}{N_2}\right]-n\cdot\left[1-E_1+\alpha_2E_1(1-\frac{N_1}{N_2})\right]$$
$$=\frac{\dot N_1}{N_1}\cdot\left[1-\alpha_1E_1+\alpha_2E_1\frac{N_1}{N_2}\right]-n\cdot\left[1-E_1+(1-\alpha_1)E_1+\alpha_2E_1\frac{N_1}{N_2}\right]$$
$$=\left[1-\alpha_1E_1+\alpha_2E_1\frac{N_1}{N_2}\right]\cdot(\frac{\dot N_1}{N_1}-n):=G^{-1}\cdot(\frac{\dot N_1}{N_1}-n)$$
$$=E_1\cdot\left[\alpha_1g_1+\alpha_2g_2\right]+e_{11}g_1-\gamma_1(1+e_{11})+e_{12}\cdot(g_2-\gamma_2)$$
$$=E_1\cdot\left[\alpha_1g_1+\alpha_2g_2\right]+(g_1-\gamma_1)\cdot(S_{11}-\alpha_1E_1)-\gamma_1+(g_2-\gamma_2)\cdot(S_{12}-\alpha_2E_1)$$
$$=E_1\cdot\left(\alpha_1\gamma_1+\alpha_2\gamma_2\right)-\gamma_1+S_{11}\cdot\left((g_1-g_2)+(\gamma_2-\gamma_1)\right)$$

Then we get the central expression for the relative growth rate of employment in sector 1:

$$(8)\qquad (\frac{\dot N_1}{N_1}-n)=G\cdot\left[E_1\cdot(\alpha_1\gamma_1+\alpha_2\gamma_2)-\gamma_1+S_{11}\cdot\left((g_1-g_2)+(\gamma_2-\gamma_1)\right)\right]:=G\cdot\Omega$$

Here we concentrate on the term called G which has an interpretation as the gains from migration or the pure reallocation gain of labour (“overflyttingsgevinst”): Use that fact that $\sum_j \alpha_j E_j = 1$ to get:

$$G:=\frac{1}{1-\alpha_1E_1+\alpha_2E_1\frac{N_1}{N_2}}=\frac{1}{1-\alpha_1E_1+\alpha_2E_1\frac{N_1}{N_2}}=\frac{1}{1-\alpha_1E_1+\frac{w_2N_2}{Y}E_1\frac{N_1}{N_2}}$$
$$=\frac{1}{1-\alpha_1E_1+\frac{w_2}{w_1}\frac{w_1N_1}{Y}E_1}=\frac{1}{1-\alpha_1E_1+\frac{w_2}{w_1}\alpha_1E_1}=\frac{1}{1-\alpha_1E_1(1-\frac{w_2}{w_1})}$$
$$=\frac{1}{1-(w_1-w_2)\cdot\frac{N_1}{Y}\cdot E_1}=\frac{1}{1-hE_1N_1}$$

where we have defined $h:=\frac{w_1-w_2}{Y}$ as the relative increase in income per worker that moves from sector 2 and gets a job in sector 1. Suppose that $w_1>w_2$ and that some migration from sector 2 to sector 1 takes place. How can we understand this gain from migration?

Suppose we have some initial increase in sector 1's employment equal to $\Delta N_{1,(1)}=\Omega N_1$, in the first step; the primary increase. The absolute increase in the number of people migrating in the first place is proportional to the initial level of $N_1$. This reallocation will produce a higher income as given by $\Delta Y_{(1)}=h\Delta N_{1,(1)}=h\Omega N_1$, which will "next" induce an increased demand for good 1 as given by $\Delta C_1 = E_1 \cdot \Delta Y_{(1)} = E_1 \cdot h\Omega N_1$. As long as there is a linear relationship between output and labour, employment will increase at a rate proportional to initial employment ; i.e. $\Delta N_{1,(2)} = (E_1 \cdot h\Omega N_1)N_1$. This is the secondary effect. Then because more people has been reallocated to the higher-wage sector, there is again an additional relative increase in income, as given by $\Delta Y_{(2)} = h\Delta N_{1,(2)} = h \cdot (E_1 \cdot h\Omega N_1)N_1$, producing an additional demand for commodity 1, as given by $E_1 \cdot h \cdot (E_1 \cdot h\Omega N_1)N_1$, and also an additional increase in employment, proportional to previous employment: $(E_1 \cdot h \cdot (E_1 \cdot h\Omega N_1)N_1)N_1 = \Delta N_{1,(3)}$. Summing all these terms as stepwize increases or increases in each stage, we get a converging geometric sequence:

$$\Delta N_1 = \sum_{j=1}^{\infty} \Delta N_{1,(j)} = \Omega N_1 + (E_1 \cdot h\Omega N_1)N_1 + (E_1 \cdot h \cdot (E_1 \cdot h\Omega N_1)N_1)N_1 + ....$$
$$= \Omega N_1 \cdot \left[1 + hE_1N_1 + (hE_1N_1)^2 + (hE_1N_1)^3 + .....\right] = \Omega N_1 \cdot \frac{1}{1 - hE_1N_1} = G \cdot \Omega \cdot N_1$$
$$= \frac{dN_1}{dt} - nN_1 \Leftrightarrow \quad \frac{\dot{N}_1}{N_1} - n = G \cdot \Omega$$

The term G is the gains from transfer of labour from the low productivity sector to the high productivity sector. Migration will stop up as the successive increases in income will decline; not because wages are equalised

## 2. *A simplified version of Leif Johansen's MSG-model*

We can express some of the main ideas in the MSG-model by looking at the following version; see Leif Johansen, 1960, Rules of Thumb for the Expansion of Industries in aProcess of Economic Growth, *Econometrica*, Vol. 28, No. 2 (April), pp. 258-271.

(1) $X_i = A_i N_i^{a_i} K_i^{b_i} e^{\gamma_i t}$ $\qquad i = 1,2,...,m$

(2) $\sum_i N_i = N$ $\qquad$ (Full employment)

(3) $\sum_i K_i = K$

(4) $X_i = N \cdot g_i(P_1,...,P_n,\frac{Y}{N})$ $\qquad i = 1,2,,...,m$

(5) $a_i = \dfrac{WN_i}{P_i X_i}$ $\qquad i = 1,2,\ldots,m$

(6) $b_i = \dfrac{QK_i}{P_i X_i}$ $\qquad i = 1,2,\ldots,m$

(Incorporated in the demand functions (4) is the budget constraint: $Y = \sum_i P_i X_i$.

Notation and symbols should be known.)

We assume CRS; $a_i + b_i = 1$. $W$ is the wage, and $Q$ is the rate of return to real capital. (We have $Q = P_K(r+\delta)$ as the user cost of capital, where $r$ is the rate of interest, $\delta$ rate of depreciation and $P_K$ is the price of capital.)

Let N and K be exogenous and let W be a constant. Then the model has $4m+2$ equations and the same number of endogenous variables, as given by:
$X_1,\ldots,X_m, N_1,\ldots.,N_m, K_1,\ldots,K_m, P_1,\ldots,P_m, Q, \dfrac{Y}{N}$.

Let us try to see how the inclusion of capital in the multi-sectoral growth model might affect or modify the conclusions we have derived from simpler models.

Use small letters for rate of changes; like $x_i := \dfrac{\dot{X}_i}{X_i}$ etc. We then have:

(1) $x_i = a_i n_i + b_i k_i + \varepsilon_i$

(2) $\sum_i \dfrac{N_i}{N} \cdot n_i = n$

(3) $\sum_i \dfrac{K_i}{K} \cdot k_i = k$

(4) $x_i = n + \sum_j e_{ij} \cdot p_j + E_i \cdot y$

(5) $p_i + x_i = w + n_i = n_i$

(6) $p_i + x_i = q + k_i$

Here: $4m+2$ equations between the same number of variables: $(x_1,\ldots,x_m, n_1,\ldots.,n_m, k_1,\ldots,k_m, p_1,\ldots,p_m, q, y)$. This system can in principle be solved. Let us assume, intuitively ok, that $k > 0 \Rightarrow q < 0$; when stock of capital increases, the relative rate of return is expected to decline. (A higher capital intensity will lower the return on capital.) Our production structure implies technical complementarity, as $n > 0 \Rightarrow q > 0$. Then we can assume that if capital grows faster than labour, as being assumed, $k - n > 0$, then $q < 0$. What about the growth in income per capita? We have $\frac{Y}{N} = Q\frac{K}{N} + W$; from which we see that if $\frac{K}{N}$ increases, then Q will decline. More capital per worker will increase income per worker, but capital becomes cheaper relative to labour. However, we **assume** that $k - n > 0 \Rightarrow y > 0$, even if $q < 0$.

We can use the equations above to find explicit expressions for $p_i$ and $x_i$. Use (5) and (6) in (1), to give:

$$(I) \quad x_i = a_i(p_i + x_i) + b_i(p_i + x_i - q) + \varepsilon_i$$

Because of CRS, we get from (I): $p_i = b_i q - \varepsilon_i < 0$. The commodity prices will decline relative to wage.

Next question: In what sectors will the price decline at a higher rate?

- Sectors that are capital intensive (high $b_i$), like manufacturing, will experience a higher decline in prices than less capital intensive sectors, like service industries. The prices will change in favour of those goods being produced in capital intensive sectors. We also believe that the rate of technical progress is higher in manufacturing than in service provision; hence this effect will reinforce the impact from capital intensity.

The demand structure will have an impact on *q*: We have from (4):

$$(II) \quad x_i = n + \sum_{j \neq i} e_{ij} \cdot p_j + e_{ii} \cdot (b_i q - \varepsilon_i) + E_i \cdot y$$

A sector producing a commodity with high income elasticity, will cause (cet.par.) $x_i$ to be high as well. We have $e_{ii} < 0, q < 0$ and $b_i > 0$, making $e_{ii} \cdot b_i \cdot q > 0$. The impact on the development of output produced when capital per worker increases, will be higher

- The higher is $b_i$

- The stronger demand for the good is affected by own price change; high $|e_{ii}|$ or highly elastic demand
- When $-q$ is large

(Example: Durables with high potentiality for technical progress and highly elastic demand.)

In addition we have a term $(-e_{ii} \cdot \varepsilon_i) > 0$, which reinforces the preceding factor, in the sense that: The role of technical progress on the rate of increase in output is more significant the more price elastic is demand.

When we have solved for $p_i$ and $x_i$, we can in CRS, solve for $n_i = p_i + x_i$:

$$n_i - n = \sum_{j \neq i} e_{ij} \cdot p_j + e_{ii} \cdot (b_i q - \varepsilon_i) + E_i \cdot y + b_i q - \varepsilon_i$$
$$= b_i \cdot q(1-(-e_{ii})) - \varepsilon_i \cdot (1-(-e_{ii})) + E_i \cdot y + \sum_{j \neq i} e_{ij} \cdot p_j$$

from which we can then draw the following conclusions, some of which support those derived earlier in Vislie, and some new: Suppose cross price effects are negligible (the last term vanishes), and $q < 0$; then we have:

- With high technical progress, elastic demand and high income elasticity, will normally produce $n_i - n > 0$.
- With high technical progress, inelastic demand and not too high income elasticity, then $n_i - n < 0$.
- The Simon effect is identified; $n_i - n$ is positively correlated with $E_i$.
- There is positive correlation between elasticity of demand, $(-e_{ii})$ and $n_i - n$.
- The relation between capital intensity (the magnitude of $b_i$) and $n_i - n$ will depend on whether the good being produced is elastic or inelastic in demand.