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ECON 4350 – Growth and Investment, fall 2011

Problem set 1 (seminar 1) – September 19

Problem 1.

First a hint: In some of the models we consider we solve rather simple differential equations, which perhaps are not known well by all.

In Domar's model we derived the differential equation for investement, as given by $\dot{I}(t) = svI(t) \Leftrightarrow \frac{\dot{I}(t)}{I(t)} = sv$. Now, because $\frac{d}{dt}(\ln I(t)) = \frac{\dot{I}(t)}{I(t)} = sv$, we find by integrating: $\int \frac{d}{dt} \ln I(t)dt = sv\int dt + const$. Call the constant for M. By definition we then have: $\int \frac{d}{dt} \ln I(t)dt = \ln I(t)$; hence $\ln I(t) = svt + M$. Furthermore, from the definition of e, it follows that $e^{\ln x} = x$. Then: $e^{\ln I(t)} = e^{svt+M} = e^M e^{svt} = I(0)e^{svt}$, with $M = I(0)$. This type of differential equations is found several places in the literature.

Now to the problem:

Consider the "knife-edge"-model presented in Lecture #2:

(1) $X(t) = Min\left\{vK(t), uN(t)\right\}$

(2) $\dot{K}(t) = sX(t)$

(3) $\dot{N}(t) = nN(t)$

Suppose that $n > sv$, so output will in the long run be constrained by available capital; $X(t) = vK(t)$. Because we get permanent unemployment, let $L(t) = \frac{v}{u}K(t)$ be the actual use of labour. Suppose we have initially have $K(0) = \frac{u}{v}N(0)$.

i) Solve for $K(t)$.

ii) Show that $\frac{L(t)}{N(t)} \to 0$ as $t \to \infty$.

iii) What will the development look like if $n < sv$? Now output is labour constrained, with actual use of capital $\hat{K}(t) = \frac{u}{v}N(t)$ and $X(t) = uN(t)$ with savings as given by (2). Show that capital equipment in the economy at any point in time will be given by $K(t) = \frac{su}{n} N(0)e^{nt} + \frac{u}{v} N(0)(1 - \frac{sv}{n})$ and that the capital utilization ratio $\frac{\hat{K}(t)}{K(t)} \to \frac{n}{sv} < 1$ as $t \to \infty$.

iv) Compare the limiting results under ii) and iii).

v) Derive a differential equation for capital per worker when $sv > n$, when we have $x(t) := \frac{X(t)}{N(t)} = Min\left\{v\frac{K(t)}{N(t)}, u\right\} := Min\left\{vk(t), u\right\}$. Show that in steady state capital is redundant.

Problem 2.

The basic neoclassical growth model is given by (notation as in the lecture):

(1) $X(t) = F(K(t), L(t))$
(2) $\dot{K}(t) = sX(t)$
(3) $\dot{L}(t) = nL(t)$
(4) $F_L(K, L) := \frac{\partial F(K, L)}{\partial L} = \frac{w}{p}$

The production function is twice differentiable, strictly increasing, strictly concave and exhibits CRS. Hence we have $F_{LL} := \frac{\partial^2 F}{\partial L^2} < 0$.

i) Show that labour demand is declining in real wage.

ii) Because F is CRS, we have $F(\lambda K, \lambda L) = \lambda F(K, L)$. Use this to show that $F_L L + F_K K = F$. (Hint: Differentiate $F(\lambda K, \lambda L) = \lambda F(K, L)$ with respect to λ, and then evaluate the function for $\lambda = 1$.

iii) Show then that each marginal productivity is homogeneous of degree zero. (Hint: Partial differentiation (say) with respect to K of $F(\lambda K, \lambda L) = \lambda F(K, L)$.) Can this property be used to derive

$F_{KL}L + F_{KK}K = 0 = F_{LL}L + F_{LK}K$? What sign must F_{LK} then have? How will this property affect the relationship between labour demand and capital equipment?

iv) We have $f(k) = F(\frac{K}{L},1) = x$, where $F(K,L)$ has the properties as outlined above, in addition to $F(0,L) = F(K,0) = 0$, the Inada-conditions hold: $\lim_{L\to 0} F_L = \lim_{K\to 0} F_K = \infty$, and $\lim_{L\to\infty} F_L = \lim_{K\to\infty} F_K = 0$. Show then that $F_K = f'(k)$, $F_L = f(k) - kf'(k)$, $F_{KK} = \frac{1}{L}f''(k)$, $F_{LL} = \frac{k^2}{L}f''(k)$ and $F_{KL} = F_{LK} = -\frac{k}{L}f''(k)$. Explain the following properties:

$f(0) = 0,\ f'(k) \in \left(0, \frac{f(k)}{k}\right), f''(k) < 0$ and $f'(0) > \frac{n}{s}$.

v) Show that the growth rate of total output can be expressed as

$$\frac{\dot{X}(t)}{X(t)} = \frac{\dot{L}(t)}{L(t)} + \frac{k(t)}{f(k(t))}f'(k)\frac{\dot{k}(t)}{k(t)} = n + \frac{\frac{K(t)}{L(t)}}{\frac{X(t)}{L(t)}}F_K = n + \varepsilon_K \frac{\dot{k}(t)}{k(t)},$$ where

$\varepsilon_K = \frac{K}{X}F_K$ is the marginal elasticity of F with respect to K.

vi) Explain why $\frac{\dot{K}(t)}{K(t)} > \frac{\dot{X}(t)}{X(t)} > \frac{\dot{N}(t)}{N(t)} = n$ in the transition process towards steady state when capital per worker is increasing.

vii) What is the relationship in steady state between the saving rate and the wage rate, on the one hand, and the saving rate and the rate of return on capital, on the other?

viii) Golden rule of accumulation is characterized by a saving rate so as to maximize steady-state consumption per worker. The implied capital per worker is given by $f'(k^{GRA}) = n$. Derive the capital income and labour income in this state, when factors of production are paid according to their marginal productivity. (Note that in GRA $\frac{C}{L} = f(k) - nk$ is maximized; hence $C^{GRA} = Lf(k^{GRA}) - Lk^{GRA}f'(k^{GRA})$.) Is there any relationship to the classical growth model?