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ECON 4350 – Growth and Investment – Fall 2011
Problem Set 3 (seminar 3) – October 10

**Question 1**

In this problem we consider the time path or development of government debt, and the ability to repay, including interst payment.

Suppose that the government borrows a fraction $\alpha$ of domestic output which is assumed to increase at a rate $g$ per unit of time. For simplicity suppose the economy is closed. If $R$ is the domestic output and $D$ government debt, private lenders receive interest payment per unit of time equal to $U = iD$, where $i$ is the rate of interest. Our assumptions then say that: $\dot{D}(t) = \alpha R(t)$, and $\dot{R}(t) = gR(t)$. If production takes place only in the private sector, private disposable income will be $Y = R + iD$, which is the tax base. A measure of the burden due to the government's payment of interest is then $\gamma := \frac{U}{Y}$, and a measure of indebtedness is $\lambda := \frac{D}{R}$, which has got a lot of attention during the European crisis.

The following questions:

- How will $\lambda$ evolve over time?
- What is the time path of $\gamma$?
- What role does $g$ play? What is the impact of a higher $i$?

**Question 2**

Consider a simple extension of the neoclassical model where we introduce *human capital* $H(t) = A(t)L(t)$, with $L$ as the labour force, and $A$ a measure of the efficiency of the labour force. (The productive capacity of the labour force is given by the amount of human capital $H$.) Assume that output is derived from the CRS production function of Cobb-Douglas type; $X(t) = [K(t)]^{1-a}[H(t)]^{a}$, where $K(t)$ is the amount of physical or real capital at *t*. A proportion $s_K$ of income at *t*, $X(t)$, is invested in real capital, and a proportion $s_H$ is invested in human capital; i.e. we have $\dot{K}(t) = s_K X(t)$ and $\dot{H}(t) = s_H X(t)$, when any type of depreciation is ignored.

a) What ratio of physical to human capital will be achieved in equilibrium where both investments yield the same return or marginal productivity?

b) Can you make any inference about the saving rates in equilibrium?

c) What about the growth rate?

**Question 3**

a) Suppose we have a macro production function $X = F(N, K)$ which has positive marginal productivities and exhibits decreasing returns to scale. Let $\dot{N}(t) = nN(t)$ and $\dot{K}(t) = sX(t)$.What is the condition for non-negative growth in income per capita?

b) Suppose next that there is some disembodied technological progress as expressed by the "new" production function $\Phi(N, K, t)$, where we define $\mu(N, K, t) := \frac{\Phi_t}{\Phi}$ as the rate of technological progress per unit of time, with $\Phi_t := \frac{\partial \Phi}{\partial t} > 0$. Derive a condition for growth in income per capita in this case.

c) We now have "land" as a third factor of production, capturing "natural resources" in a wide sense, available in a fixed size, $R_0$, but with some land-augmenting technological progress at a rate $m \geq 0$. Assume a Cobb-Douglas production structure, $X(t) = (N_0 e^{nt})^a (K(t))^b (\mathrm{R}_0 \mathrm{e}^{mt})^c$. You can assume that $a + b + c = 1$. What is the condition for a "semi-balanced" growth, with $\frac{\dot{X}}{X} = \frac{\dot{K}}{K} = g$, in this case? What is the growth rate in income per capita?

d) Suppose that the natural resource used in the (CRS) production function is the flow of depleted exhaustible resources $R(t) = -\dot{S}(t)$, where $S(t)$ is the stock of resource available at t. (We have: $S(t) = S_0 - \int_0^t R(\tau) d\tau$, with $S_0$ as the initial stock.) Let $X(t) = e^{\lambda t} (N_0 e^{nt})^a (K(t))^b (R(t))^c$, where $e^{\lambda t}$ is a Hicks-neutral (output-augmenting) technological progress term. Suppose we have a constant rate of resource depletion $\delta$ so that $R(t) = \delta S_0 e^{-\delta t}$ for any t. Derive the growth rate $g = \frac{\dot{X}}{X} = \frac{\dot{K}}{K}$ in this case.

e) If you want to do more: Solve for the saving rate (or the depletion rate) when requiring efficiency which (can be shown; see below) is sustained if the marginal productivity of the resource grows at a rate equal to the marginal productivity of capital: $\frac{d}{dt}\left[c\frac{X}{R}\right] = b\frac{X}{K}$ or $\frac{\dot{X}}{X} + \delta = b\frac{g}{s} \Leftrightarrow g + \delta = b\frac{g}{s}$. Use $g$ derived earlier to solve for the saving rate or for the depletion rate. What is the relationship between $s$ and $\delta$?

(A remark on efficiency: Suppose that we have an economy producing a commodity that can be used for consumption and capital accumulation, by using capital and an exhaustible natural resource. Efficiency must now be interpreted in a dynamic sense as: Let $C(\cdot)$ be a feasible consumption path on $\left[0,\infty\right)$, obeying $C(t) = X(t) - \dot{K}(t)$, where output follows from the production function, $R(t) \geq 0$, $K(t) \geq 0$ and with a given $K(0) = K_0$. A consumption path $\hat{C}(\cdot)$ is called efficient if this path is feasible, and we **cannot** find another path $C^0(\cdot)$, so that $C^0 \geq \hat{C}$ for all t, and strict inequality for some interval $\left[t', t''\right]$, $0 \leq t' < t''$. If that is the case, the following arbitrage condition must hold. Let marginal productivity of capital at t be $q(t)$ and the marginal productivity of natural resource at t be $Q(t)$. To characterize an efficient consumption path, we can think of two ways of increasing consumption at $t+h$: Higher consumption can be generated by using a unit more of resources at $t+h$. This will generate $Q(t+h)$ more units available for consumption at $t+h$. Another option is to increase the use of resources at $t$ by one unit to produce $Q(t)$ more units of the output at t, and then use this additional output for investment to provide an increase in the capital stock of $Q(t)$ over the interval $\left[t, t+h\right]$. During this interval, with $Q(t)$ more capital, an amount equal to $h \cdot q(t)$ of additional output can be produced **per unit capital**. When also taking into account that capital itself can be consumed, we have now available for consumption: $Q(t) + hq(t)Q(t) = [1 + hq(t)]Q(t)$. For an efficient consumption path, one has to be indifferent between these options; hence: $Q(t+h) = [1 + hq(t)]Q(t) \Leftrightarrow \frac{Q(t+h) - Q(t)}{h} = q(t) \cdot Q(t)$. Suppose $h$ is small, and take limits when assuming differentiability, we get: $\dot{Q}(t) = q(t) \cdot Q(t) \Leftrightarrow \frac{\dot{Q}(t)}{Q(t)} = q(t)$, saying that along an efficent consumption path the relative rate of change in the marginal productivity of natural resources, $\frac{\dot{Q}(t)}{Q(t)}$, must equal the marginal productivity of capital, $q(t)$.)