ECON 4350: Growth and Investment
Lecture note 7

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7 Endogenous saving, the Ramsey model

Required reading: BSim: 2 (except 2.6.7 and 2.7)
Supplementary reading on theory of optimal control: Obstfeld (1992)
The classics: Ramsey (1928), Cass (1965), Koopmans (1965)

7.1 Why make savings behavior endogenous?

- Before we start our analysis of how to make savings behavior in the Solow-model endogenous, it is worth to reconsider the reasons for doing this.
  - A natural generalization
  - It gives richer comparative statistics. I.e. facilitates analysis of how the economy reacts to changes in e.g. interests rates, tax rates.
  - It gives a richer description of the transitional dynamics.
  - Allows for normative assessments of policy.
- In particular, we should remember that the Solow-model primarily tells us something interesting about growth in the short run (i.e. the phase with transitional dynamics). In addition, the neo-classical revival seem to suggest that the short run is 'long', meaning that adjustment to the steady-state takes a long time.
- Both of these factors calls for more detailed modeling of this dynamics.

7.2 Profit-maximizing firms and utility-maximizing households

7.2.1 Households

- H identical households. That is, each household is of size $L(t)/H$.
- Each household grows with exogenous rate n (identical to the growth rate of population).
- Households are extended across generations (dynasties).
- Since all household are identical we can normalize $H = 1$, i.e. consider a representative household. For convenience we set $L(0) = 1$, so

$$L(t) = e^{nt}$$

Household preferences

- Current household members (acting as one entity) seek to maximize:

$$U = \int_{t=0}^{\infty} u(c(t))e^{nt}e^{-\rho t}dt \tag{1}$$

  Where $c(t)$ equals the consumption level of *each* household member (assumed identical across household members).

- The so-called felicity function $u(c)$ characterizes the utility of a given household member of consuming c at any given point in time.
- Since $L(t)$ is the number of members in the household, and $c(t)$ is the consumption of each member, the household's utility function is additive in the utility of the members (both present and future members). Further, it is additively separable over time for each individual.
- The ρ equals the constant (subjective) discount rate, i.e. it reflects the impatience of the household members. It is *also* identical to the factor of discounting of felicity across generations, i.e. that current household members discount the felicity of future generations relative to their own.
- Note that this formulation both implies a strong form of altruism within the family dynasty, and a limited form of cardinal utility.
- Households have perfect foresight. I.e. they can deduce what the future will look like, and there is no uncertainty.

- We assume that $\rho > n$, so that U is well defined (finite if c is constant over time).

- The felicity function $u(c)$ satisfies $u'(c) > 0$ and $u''(c) < 0$. The concavity reflects a desire for smooth consumption over time.

The household budget

- Households hold assets in the form of capital or as loans. Each household member supplies inelastically one unit of labor.

- Household are price-takers, i.e. take as given the path $\{r(t), w(t)\}$ of prices .

- In equilibrium all markets clear, so there is no idle labor or capital.

- We hence have

$$\frac{d(\text{Assets})}{dt} = r \cdot (\text{Assets}) + wL - cL$$

or with $a = \text{Assets}/L$ (assets per household member)

$$\dot{a}(t) = w(t) + r(t)a(t) - c(t) - na(t) = w(t) - c(t) + a(t)(r(t) - n) \quad (2)$$

- Note that this is a *dynamic* budget constraint, i.e. one that holds continuously for all t.

- We must rule out that households can accumulate debt forever. If this is not the case, it will obviously be optimal to pay for both consumption and interest on present debt by borrowing yet more money. We therefore require that

$$\lim_{t\to\infty} a(t)e^{-\int_0^t [r(v)-n]dv} \geq 0 \quad (3)$$

i.e. that the net present value of assets is asymptotically non-negative. This is often referred to as a condition ruling out Ponzi-games. In reality, we would expect stricter conditions on credit.

Utility maximization

- The household faces the problem of optimizing the utility (1) subject to the dynamic budget constraint (2) and the Non-Ponzi game condition (3). (Formally it is also subject to $c(t) \geq 0$ and $a(0) = a_0$).

- This problem falls within a class of problems best studied by the theory of optimal control. It is also possible to solve it without using this theory. The alternative approach is, however, cumbersome and makes it difficult to reformulate the problem to cover modified situations.

- In this course we therefore make use of the main results from the theory of optimal control. It is quite straightforward to learn to apply the procedure for solving the type of problems we encounter, and nothing more will be expected of you. Even if you find it hard to really understand the method fully, you should find comfort in the fact that the results we end up with are quite easy to interpret directly.

- We will cover the method in more detail on the next seminar. See also e.g. BSiM A.3.

- We start by constructing the (present value) Hamiltonian for the problem:

$$J = u(c(t))e^{-(\rho-n)t} + \nu(t)[w(t) + (r(t) - n)a(t) - c(t)] \qquad (4)$$

  This is pretty much like the Lagrangian in static optimization, and $\nu(t)$ takes the role of the multiplier. Note that the first part of the expression is the instantaneous utility (at time t) and the second term is the 'multiplier' times the dynamic constraint. Changes in either the payoff-function or the budget are then easy to include in the Hamiltonian.

- The solution to the problem is then characterized by the following three conditions:

$$\frac{\partial J}{\partial c(t)} = 0 \qquad (5)$$

$$\dot{\nu}(t) = -\frac{\partial J}{\partial a(t)} \qquad (6)$$

$$\lim_{t \Rightarrow \infty} [\nu(t) \cdot a(t)] = 0 \qquad (7)$$

  The first one is familiar and basically the same as the Maximum principle for the Lagrangian. The second, the Euler equation, is perhaps less intuitive and characterizes the dynamic dimension. From the envelope theorem we know that the derivative of the value function with respect to a parameter is the derivative of the Lagrangian with respect to this parameter. Thus we can think of $\frac{\partial J}{\partial a(t)}$ as the shadow price on increasing the budget. The third condition is called the transversality condition. It basically says that it will not be optimal to end up ('at infinity') with keeping net assets if these are valuable. The transversality condition can often be ignored in 'sloppy' applications of the method.

- The two first conditions are in many ways most central for characterizing the solution. The transversality condition is also crucial, but in a more formal way that is not really that interesting in our applications. We will therefore be somewhat sloppy and often ignore this aspect of the solution (in a loose analogy this is the same as ignoring economically implausible corner solutions in static optimization). You should, however, study the discussion of the transversality condition in BSiM.

- We therefore get

$$\frac{\partial J}{\partial c(t)} = 0 \Rightarrow \nu = u'(c)e^{-(\rho-n)t} \tag{8}$$

$$\dot{\nu}(t) = -\frac{\partial J}{\partial a(t)} \Rightarrow \dot{\nu} = -(r-n)\nu \tag{9}$$

$$\tag{10}$$

or by differentiating the first condition and inserting for ν we get the Euler equation

$$r = \rho + \left[-\frac{u''(c)\cdot c}{u'(c)}\right]\cdot\frac{\dot{c}}{c}$$

where the term in the brackets is the inverse of the intertemporal elasticity of substitution.

- The equation tells us the following: Households choose consumption so that they equate the rate of return of saving (r) to the rate of return to consuming today instead of later (equal to the discount factor ρ offset by the rate of decrease of marginal utility of consumption due to growing c).

- It is common practice to work with the special felicity function of the form

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \quad \theta > 0 \tag{11}$$

For this function the intertemporal elasticity of substitution is $1/\theta$, for this reason BSiM refer to it as the *constant intertemporal elasticity of substitution* (CIES) utility function. However, it is more frequently referred to as the CRRA (Constant Relative Risk Aversion) utility function, since it also has constant relative risk aversion (equal to θ).

- In this case the Euler-equation becomes

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\theta} \tag{12}$$

- The intuition now becomes more transparent: Consumption (per A) is increasing when the real rate of return is higher than the rate at which the household discounts future consumption (in this situation one is saving more and hence consumption *grows* faster). But the smaller is θ the more one is willing to substitute consumption intertemporally to exploit the difference between $r(t)$ and ρ.

7.2.2 Firms

- Many equal firms with technology characterized by a neo-classical production function $Y = F(K, TL)$.
- Again we consider a representative firm.
- The technology parameter T follows an exogenous path, growing at rate x, we normalize $T(0) = 1$.
- Firms rent capital and hire labor from households.
- Firms are price-takers in the markets for the inputs to production, that is, they adjust to given paths $\{r(t), w(t)\}$.
- Firms have perfect foresight. Profit-maximizers.
- Since capital and loans are perfect substitutes as stores of value, we must have $r = R - \delta$ or $R = r + \delta$.
- A representative firm's profit at any given point in time is

  $$\pi = F(K, TL) - (r + \delta) - wL$$

  or transforming to units per efficient worker

  $$\pi = TL[f(\hat{k}) - (r + \delta)\hat{k} - we^{-xt}]$$

- Since there are no adjustments costs (i.e. what it does today does affect what it can do/costs later) the firm will seek to maximize profits at any given point in time.

- We then have the usual first order conditions

$$\begin{aligned} f'(k) &= r+\delta & (13) \\ w &= [f(\hat{k})-\hat{k}f'(\hat{k})]e^{xt} & (14) \end{aligned}$$

and there is no profit in equilibrium (for all values of TL).

7.3 Equilibrium

- In equilibrium all debt must cancel, so $a = k$.
- Inserting $a = k$ and the factor prices in the household's budget constraint then gives us (after converting to units per efficient worker)

$$\dot{\hat{k}} = f(\hat{k}) - \hat{c} - (x+n+\delta)\hat{k} \tag{15}$$

- This is basically the same fundamental equation for the capital intensity as in the Solow model (the reason we end up with this equation even in the presence of profit-maximizing firms is the same as before).
- The difference is that the term for gross saving is now $f(\hat{k}) - \hat{c}$ instead of $sf(\hat{k})$ (note that $f(\hat{k}) - \hat{c}$ implies $s(\hat{k})f(\hat{k})$, i.e. a savings rate that varies with $\hat{k}$).
- Thus we must also characterize the behavior of $\hat{c}$ in order to have a full description.
- This follows from the Euler-equation, which after inserting for $r(t)$ and converting to units per efficient worker, reads

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \frac{\dot{c}(t)}{c(t)} - x = \frac{f'(\hat{k}) - \delta - \rho - \theta x}{\theta} \tag{16}$$

- The two equations (15) and (16), together with the initial condition $\hat{k}(0)$ and the transversality condition determines the time paths of $\hat{k}$ and $\hat{c}$, which is what we need to fully characterize the economy.

- We draw in the $\dot{\hat{k}} = 0$ and $\dot{\hat{c}} = 0$ lines in a phase-diagram in $(\hat{k}, \hat{c})$. This shows the existence of a saddle-path and a steady state.

- We can use the phase diagram to do comparative statistics. Note that we can not get only get gradual adjustments along the $\hat{k}$ axis (no jumps). (Unless the exogenous shifts affect $\hat{k}(t)$ directly).

7.4 Long run and the Solow-model

- The main conclusion concerning the long run is the same as before: We reach a steady state with $\dot{\hat{k}} = 0$ and $\dot{\hat{c}} = 0$.

- Thus, from a descriptive point of view the Ramsey model has not taught us much more about what happens in the long run.

References

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