A note on Harrod’s growth model

As opposed to Domar’s planning model where one derived the required rate of increase in investment so as to have permanent, full capacity utilisation, Harrod considered in his growth model from 1939 a decentralised capitalistic market economy. This model is said to be the first contribution to the literature on modern economic growth. The model is difficult to grasp but not less important as it combines actively investment behaviour among entrepreneurs and their expectations about future demand. In this short note I will present the main ideas, and show the implications of the inherent instability in Harrod’s economy, known as the “knife-edge-problem”. The model is a medium-term demand-dominated model (inspired by the Keynesian view), as opposed to the neoclassical long-term supply-dominated growth model. Another important feature is that Harrod assumed a rigidity in the production structure (Leontief-technology), as was supposed by the classical economists, whereas the neoclassical model assumed substitution between labour and capital.

Contrary to the classical approach, both population dynamics and the saving rate were assumed to be exogenous in Harrod’s model.

The inherent instability in the Harrod-model is a necessary way out of a mathematical model with too many restrictions imposed on the variables. As we will show, balanced growth with labour and capital growing at the same rate will occur only by pure chance, because all factors determining growth will be exogenous. This instability manifested itself as a “knife-edge-growth-path”.

His model has the following features:

- Saving behaviour – a fixed (exogenous) saving rate.
- Leontief-technology, with fixed production coefficients, but with the possibility of labour-saving technical progress; so-called “Harrod neutral technical progress”. (Here we drop technical progress.)
- Expectations governing investment decisions.
- Exogenous population dynamics; annual rate of growth equal to 100n%; called the “natural” rate of growth.
- Discrete time.

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Question: What are the stability properties of the model? Does the model exhibit a stable steady-state process?

Investment demand is determined from an investment function of the accelerator type, with the constant capital-output ratio, \( \frac{1}{v} \), serving as an accelerator. Expected increase in effective demand will determine planned investment, which, through the multiplier, produces a certain level of effective demand in the economy.

Let expected effective demand in period \( t \) be given by the investors’ beliefs or expectations about future demand, as given by, \( \hat{X}_t \), whereas realised output in the preceding period is \( Y_{t-1} \). (Longer lags are possible, but it will only obscure the analysis.) Hence investment demand is given by: \( I_t = \frac{1}{v} [\hat{X}_t - Y_{t-1}] \), which will determine realised output in period \( t \), through a standard Keynesian multiplier.

\[
\frac{1}{1 - c} = \frac{1}{s}, \text{ according to } Y_t = \frac{1}{s} I_t, \text{ where } s = 1 - c, \text{ and } s \text{ is the constant saving rate, while } c \text{ is a constant marginal propensity to consume, with an underlying consumption function } C_t = c Y_t. \text{ (Other demand components, like public spending, have been ignored.)}
\]

Some notation:
Define the ratio of actual to expected demand in period \( t \), the expected rate of growth, \( \hat{g}_t \), and the warranted rate of growth, \( G \):

\[
(1) \quad \frac{Y_t}{\hat{X}_t} = \frac{1}{s} \cdot \frac{1}{s} \cdot I_t = \frac{1}{s} \cdot \frac{1}{s} \cdot \frac{1}{v} [\hat{X}_t - Y_t] = \frac{1}{sv} \left[ \frac{\hat{X}_t - Y_{t-1}}{X_t} \right] = \frac{1}{sv} \cdot \hat{g}_t
\]

where the expected growth rate is calculated as a proportion of expected output in period \( t \). The novelty introduced by Harrod was the concept “warranted” rate of growth or a “moving equilibrium growth path”. This growth rate is the “correct” one in the sense that this rate of growth will create demand according to what was expected.

It follows directly that expectations will be realised, \( \hat{X}_t = Y_t \), if and only if we have

\[
(2) \quad \hat{g}_t = sv := G
\]

where \( G := sv \) is the “warranted” rate of growth, which is realised if expected. Along the warranted growth path, expectations are fulfilled. If we expect the economy to grow at the rate \( sv \), the economy will grow at this rate. The warranted rate of growth will leave the entrepreneurs convinced that they have made the correct investment
decisions: They have installed additional capital equipment necessary to produce the additional expected output, which is realised.

If we expect the economy to grow at the rate $G$, capital will grow at the same rate, because we have a linear production technology. If we at the same time have $G = \frac{sv}{g}$, the labour force will grow at the same rate. In this case, we have a dynamic equilibrium with balanced growth, neither unemployment nor underutilisation of capital. Both inputs are fully employed, and expectations are realised as we move on. This is a steady-state equilibrium path, with realised growth rate as a proportion of the output in the final period, given by $g_t \equiv \frac{Y_t - Y_{t-1}}{Y_t} = sv = G$. If we have $sv = n$, the output path will be compatible with expected output path. If the entrepreneurs happen to have expectations compatible with what is actually realised, then the equilibrium growth rate will be realised for all periods.

But of course, there is no logical reason why the two growth rates should become equal! The two sides are determined independently of each other. All terms in the condition for balanced growth, $sv = n$, are exogenous.

What if they are not equal! What will be the outcome if $\bar{g}_t \neq sv = G$? How will the economy develop if entrepreneurs have expectations different from what can support a dynamic equilibrium? What will happen if they are too optimistic or too pessimistic? On using the definition of the actual growth rate, $g_t$, together with (1), we find:

\begin{equation}
(3) \quad g_t = \frac{Y_t - Y_{t-1}}{Y_t} = 1 - \frac{Y_{t-1}}{X_t} = 1 - \frac{\bar{X}_t}{Y_t} \cdot (1 - \bar{g}_t) = 1 - \frac{1 - \bar{g}_t}{\bar{g}_t} \cdot sv
\end{equation}

or

\begin{equation}
(3)’ \quad \frac{1 - g_t}{1 - \bar{g}_t} = \frac{sv}{\bar{g}_t}
\end{equation}

We then have, except for the case analysed above: $g_t = \bar{g}_t \iff \bar{g}_t = sv$:

If $\bar{g}_t > G = sv \quad \text{then} \quad \frac{1 - g_t}{1 - \bar{g}_t} < 1 \quad \text{and} \quad g_t > \bar{g}_t$

If $\bar{g}_t < G = sv \quad \text{then} \quad \frac{1 - g_t}{1 - \bar{g}_t} > 1 \quad \text{and} \quad g_t < \bar{g}_t$
This is the source to Harrod’s instability problem. If the entrepreneurs expect a growth rate above the warranted rate of growth (i.e. they are optimistic), the realised growth rate will be even higher than what was expected. Ex post they may feel that they were not sufficiently optimistic!

On the other hand, if the expected rate of growth falls below the warranted rate of growth – describing some pessimism among the entrepreneurs – then realised growth rate will fall below the expected growth rate; hence ex post the entrepreneurs may feel they were not sufficiently pessimistic ex ante!

A simple illustration provided by Sen may explain this result. Suppose that \( s = 0.20 \) and \( v = 0.5 \). Then \( G = sv = 0.10 \).

Suppose that \( Y_{t-1} = 90 \), and a realised growth rate \( g = 0.1 \), defined as a 10% increase in output, as a fraction of final output, we get \( Y_t = 100 \). If the entrepreneurs expect \( \hat{X}_t = 100 = Y_t \), i.e. an expected growth rate equal to the warranted one, they will plan investment according to \( I_t = \frac{1}{v} [\hat{X}_t - Y_{t-1}] = 2 \times 10 = 20 \). These 20 units of additional capital will be necessary additional capacity to produce the additional output equal to 10 when the capital-output ratio \( v = 0.5 \). Through the multiplier, \( \frac{1}{s} = 5 \), this investment demand will generate an output exactly equal to 100. Hence, expectations are fulfilled.

Suppose next that the entrepreneurs have more optimistic expectations, say that they expect \( \hat{X}_t = 101 \). Then the desired or planned investment is determined from
\[
I_t = \frac{1}{v} [\hat{X}_t - Y_{t-1}] = 2 \times 11 = 22 ,
\]
which generates new capacity so that the economy is able to produce 11 additional units of output. From the multiplier, realised output is determined as \( Y_t = 5 \times 22 = 110 > 101 = \hat{X}_t \). The entrepreneurs may feel that they were not sufficiently optimistic. Realised growth rate is \( \frac{110 - 90}{110} = 0.1818 > 0.1 = sv \).

This will create a process of optimism, which will cause a shortage with an accompanying dehoarding of inventories and some inflation so as to make realised sales at the new prices equal to expected sales.

If entrepreneurs are a bit pessimistic, with \( \hat{X}_t = 99 \), then \( I_t = 2 \times 9 = 18 \), so as to produce additional capacity for 9 more units out the output. But realised output is determined from the multiplier, as given by \( Y_t = 5 \times 18 = 90 < 99 = \hat{X}_t \); hence \( g_t = 0 < sv = 0.1 \). In this case we will run into a depression with an increasing pessimism taking place.
Let us then turn to the other side of the model – bringing in the natural rate of growth related to the growth of the labour force. So far we have analysed the inherent instability caused by some divergence between expected and actual demand, and the impact on capital accumulation. We’ll show that we end up on a knife-edge, in the following sense: Any discrepancy between the warranted rate of growth and the natural rate of growth will cause some imbalance in the economy, characterised either by inflation (when entrepreneurs are overall optimistic), or by unemployment or depression (under pessimistic expectations).

On combining the two sides of the model, we have seen that balanced growth is possible only if $sv = n$, where we should note that all terms are exogenous. (The subsequent models of economic growth have tried to explain one or more of these terms. The classical economists had a model with endogenous population growth and endogenous saving rate.) Harrod considers the natural rate of growth as the maximum sustainable rate of growth in the long run. We’ll see why.

If this condition is not met, which should happen only by pure chance, we face a serious imbalance or a knife-edge problem. A small deviation from the equilibrium growth path where $sv = n$, will push the economy further away from the path.

Suppose we start out with some unemployment in the economy, and suppose that $g = sv > n$. Now we have a steady growth process at the warranted rate of growth, above the natural rate of growth. As the unemployed workers are absorbed in the labour force, the economy can grow according to $sv$. But sooner or later the economy will hit the full employment constraint. Labour becomes the constraining factor of production. The actual rate of growth shifts downwards to the lower natural rate of growth, i.e. to $g = n < sv$. The entrepreneurs' expectations will not be fulfilled, and this disruption may trigger off a movement towards a depression.

If $g = sv < n$, there is a growing unemployment, with no self-correcting mechanism in the model.

The knife-edge instability problem is resolved in later models by making some of elements in the equilibrium condition endogenous, as in the classical theory.

What are the policy issues in this model? What kind of internal adjustment or self-correcting mechanisms can be thought of? What about learning if expectations are not fulfilled? We can extend the model so as to incorporate some self-correcting adjustment mechanism, as being done in business cycle models by Goodwin, Hicks and Haavelmo. Unfortunately, we don’t have time to consider these adjustment issues.