Final exam – retake

2024

The exam consists of three parts. The first part is a question without any math (except, perhaps, stating an accounting equation), the second question is related to the Solow model, and the third is related to the endogenous-growth frameworks.

1 Convergence (30 points)

(a) Describe the assumptions under which the Solow model predicts "convergence". Explain what convergence means in this context.

The Solow model predicts that economies that share the same technology (production function, depreciation rate, level of technology) and the same saving behavior do converge to the same level of GDP per capita.

Convergence either means that the dispersion across countries of GDP per capita shrinks over time ("σ convergence") or that the growth rate of GDP per capita is higher for low-income countries ("β convergence").

[Either convergence concept is correct, no need to discuss both, we covered β convergence in class]

(b) Describe a regression that you can run to test whether there is convergence in GDP across regions. Explain the regression, in particular which regression coefficient is of interest and how do we interpret it?

To test (β) convergence, we run the regression

$$\text{log GDP per capita growth} \sim \alpha + \beta \text{log GDP per capita}$$

where the parameter of interest is β. If β is negative, then growth is higher for low income countries and we have (β-)convergence.

(c) In broad strokes, do we see convergence across countries? Give an example of a set of regions (e.g., countries) for which we have seen convergence.

In broad strokes, we do not. Several examples were given in class. For example, we see convergence among OECD countries.

2 Solow model (35 points)

In this problem, we characterize factor prices in the Solow model.

(a) Assume no population growth ($n = 0$) and no technological growth ($g = 0$). Write down the law of motion for capital given a savings rate s, production function $F(K, L)$, and depreciation rate δ.

Given the stated assumptions, the law of motion is given by $K_{t+1} = F(K_t, L) + (1 - \delta)K_t$.

(b) Assume that the markets for labor and capital are perfectly competitive. Write down the problem of a profit-maximizing representative firm which has access to the technology $F(K, L)$.

A profit maximizing firm solves the problem

$$\max_{K,L} F(K, L) - rK - wL.$$

(c) Show that, in equilibrium, factors (labor and capital) are paid their *marginal product*.

We take the first-order conditions of the firm's problem,

$$\begin{aligned} \partial K : \quad & F_K(K, L) - r = 0, \\ \partial L : \quad & F_L(K, L) - w = 0. \end{aligned}$$

Rearranging, we arrive at $r = F_K$ and $w = F_L$, i.e., the factors are paid their marginal product.

(d) Write down an expression for the labor share, in terms of the production function F, its derivative(s), as well as K and/or L.

The labor share of GDP is payments accruing to labor (wL) divided by GDP ($F(K, L)$),

$$\text{Labor share} = \frac{wL}{F(K, L)} = \frac{F_L(K, L)L}{F(K, L)}.$$

(e) Finally, assume that $F(K, L) = K^\alpha L^{1-\alpha}$. Show that the labor share equals $1 - \alpha$.

Using that $F(K, L) = K^{\alpha} L^{1-\alpha}$ and that $F_L(K, L) = (1-\alpha) K^{\alpha} L^{-\alpha}$, we get

$$\text{Labor share} = \frac{F_L(K, L)L}{F(K, L)} = \frac{(1-\alpha)K^{\alpha}L^{-\alpha}L}{K^{\alpha}L^{1-\alpha}} = 1 - \alpha.$$

3 Schumpeterian model (35 points)

In this exercise, we go through the basic Schumpeterian growth model. Final-good production is given by

$$Y_t = L^{1-\alpha} \int_0^1 A_{it}^{1-\alpha} x_{it}^{\alpha} di$$

where each x_{it} is the quantity of intermediate product i and A_{it} reflects the quality of the product. Each intermediate product has its own monopoly, and its price equals its marginal product in the final sector.

(a) Show that the price of good i is given by $p_{it} = \alpha(A_{it}L)^{1-\alpha} x_{it}^{\alpha-1}$.

Since the price of the intermediate good is given by its marginal product in the final sector, we have $p_{it} = \frac{d}{dx_{it}} Y_t = \alpha(A_{it}L)^{1-\alpha} x_{it}^{\alpha-1}$.

(b) The intermediate goods are produced one-to-one using the final good. Write down intermediate-good monopolist i's problem and show that the solution to the monopolist's problem is given by

$$\begin{aligned} x_{it} &= \alpha^{2/(1-\alpha)} A_{it} L, \\ \Pi_{it} &= \pi A_{it} L \qquad \text{with } \pi = (1-\alpha)\alpha^{(1+\alpha)/(1-\alpha)}. \end{aligned}$$

The intermediate-good monopolist i's problem is

$$\Pi_{it} = \max_{x_{it}} p_{it}(x_{it}) x_{it} - x_{it} \Leftrightarrow \Pi_{it} = \max_{x_{it}} \alpha(A_{it}L)^{1-\alpha} x_{it}^{\alpha} - x_{it}$$

with first-order condition $\alpha^2 (A_{it}L)^{1-\alpha} x_{it}^{\alpha-1} - 1 = 0$ which after some rearranging yields $x_{it} = \alpha^{2/(1-\alpha)} A_{it} L$.

Pluggin this into the objective yields

$$\begin{aligned} \Pi_{it} &= \alpha(A_{it}L)^{1-\alpha} \left(\alpha^{2/(1-\alpha)} A_{it} L\right)^{\alpha} - \alpha^{2/(1-\alpha)} A_{it} L = \\ &= \left(\alpha^{1+2\alpha/(1-\alpha)} - \alpha^{2/(1-\alpha)}\right) A_{it} L = \left(\alpha^{(1+\alpha)/(1-\alpha)} - \alpha^{1+(1+\alpha)/(1-\alpha)}\right) A_{it} L \\ &= (1-\alpha)\alpha^{(1+\alpha)/(1-\alpha)} A_{it} L. \end{aligned}$$

(c) In each sector, there is a single entrepreneur who spends final output in research and innovates with probability $\phi(n_{it}) = \lambda n_{it}^{\sigma}$ (with $0 < \sigma < 1$) where $n_{it} = R_{it}/A_{it}^*$ and $A_{it}^* = \gamma A_{i,t-1}$. If the entrepreneur succeeds, she becomes the monopolist for one period and $A_{it} = \gamma A_{i,t-1}$. Write down the entrepreneur's maximization problem and show that the research arbitrage equation is

$$\phi'(n_{it})\pi L = 1.$$

If the entrepreneur succeeds, she makes profits $\pi A^*_{it} L$. The maximization problem is thus

$$\max_{R_{it}} \phi(R_{it}/A^*_{it})\pi A^*_{it} L - R_{it}$$

with first-order condition $\frac{\phi'(R_{it}/A^*_{it})}{A^*_{it}}\pi A^*_{it} L - 1 = 0$ or $\phi'(R_{it}/A^*_{it})\pi L = 1$. With the notation, $n_{it} = R_{it}/A^*_{it}$, we have arrived at the sought after research arbitrage equation. In particular, the probability of a successful innovation, $\phi(n_{it})$, is equal across sectors.

(d) It can be shown that the growth rate of GDP is equal to the growth rate of aggregate technology $A_t = \int_0^1 A_{it} di$. Argue heuristically that the growth rate of GDP equals $g = (\gamma - 1)\phi(n)$ where n solves the research-arbitrage equation.

The probability for any given entrepreneur to succeed is $\phi(n)$. By the law of large numbers, the share of entrepreneurs who succeed is $\phi(n)$. In these sectors, productivity increases by a factor $\gamma - 1$. In the other sectors, productivity stays constant. The overall productivity growth is

$$g = \underbrace{(\gamma - 1)}_{\text{prod. growth if success}} \times \underbrace{\phi(n)}_{\text{share success}} + \underbrace{0}_{\text{prod. growth if no success}} \times \underbrace{(1 - \phi(n))}_{\text{share no success}}$$

which equals $(\gamma - 1)\phi(n)$.

(e) Now, assume that when an entrepreneur makes a successful innovation, with probability χ they fail to get a monopoly (e.g., someone else sees their recipe, runs to the patent office and gets the patent before them). Adjust the entrepreneur's maximization problem and the research arbitrage condition to reflect this. Qualitatively, will there be more or less research when $\chi > 0$? Qualitatively, how does the growth rate depend on χ?

Now, the entrepreneur's maximization problem is

$$\max_{R_{it}} (1 - \chi)\phi(R_{it}/A^*_{it})\pi A^*_{it} L - R_{it}$$

with foc (i.e., research arbitrage condition) $(1-\chi)\phi'(n)\pi L = 1$.

With $\chi > 0$, there will be less research (since ϕ' is monotonically decreasing in n). The growth formula is unchanged, $g = (\gamma - 1)\phi(n)$, so when n falls the growth rate falls as a result. That is, the growth rate depends negatively on χ.