Final exam

2023

The exam consists of three parts. The first part is a question without any math (except, perhaps, stating an accounting equation), the second question is related to the Solow model, and the third is related to the endogenous-growth frameworks.

1 Rival and exclusionary goods (30 points)

(a) Describe Paul Romer's distinction between rival and non-rival goods, as well as exclusionary and non-exclusionary goods. Give examples of rival and exclusionary, rival and non-exclusionary, non-rival and exclusionary, non-rival and non-exclusionary goods (the examples need not be 100% perfect but you should write 1-2 sentences explaining why each example works).

Rival goods are goods which are such that the use by one person prohibits the use by another person. By contrast, non-rival goods are goods such that the use by one person does not affect whether the same good can be used by another person.

Exclusionary goods are such that an individual can prohibit the use of the good by others. By contrast, non-exclusionary goods are such that no one can prohibit the use of the good by others. Whether a good is exclusionary or not partially depends on institutions (e.g., patent law).

Rival and exclusionary: food. If I eat a banana, you cannot eat it. Further, as long as I guard my banana, I can stop you from eating it.

Rival and non-exclusionary: fish in the ocean, absent fishing rights. Anyone can go to the sea and fish so the fish is non-exclusionary. However, for each fish I catch, I reduce the number of fish available to you. The fish is thus rival.

Non-rival and exclusionary: a recipe for a novel drug (with patents). Once a new drug has been discovered, any pharmaceutical company can reproduce the drug at low cost, the basic 'recipe' for the drug is non-rival. However, with patent laws it may be prohibited for other pharmaceutical companies to produce the drug, so the recipe is an exclusionary good.

Non-rival and non-exclusionary: a mathematical theorem such as Pythagoras' theorem. There are no property rights for basic mathematics so basic mathematics is non-exclusionary. Further, me learning about Pythagoras' theorem in no way diminishes your ability to learn about Pythagoras' theorem.

(b) Why does Paul Romer introduce the distinction between rival and non-rival goods? Relate the distinction to *increasing returns to scale* and *ideas* as the engine of growth. Make sure to define increasing returns to scale in the process.

Whereas standard economics (e.g., the first welfare theorem) deals with rival goods, Romer argues that non-rival goods are the engine of growth. This is because they allow for *increasing returns to scale*.

In standard neoclassical economics, if we double all the input (capital, labor, ...), then we double all the output. Once we take non-rival goods into account, we get increasing returns to scale: when we double all the input (capital, labor, recipes, ...), we more than double output. Why? With twice as much capital and twice as much labor, we would have produced twice as much, keeping the number of recipes constant. With an increase in the number of (e.g., drug) recipes, we more than double output.

Because of increasing returns to scale, an increase in the number of recipes, unlike an increase in capital, does not feature diminishing returns and can be an engine of growth.

2 Solow model (40 points)

In this problem, we characterize the capital-output ratio in the Solow model. Assume the population growth is n ($L_t = (1+n)^t L_0$) and the labor-augmenting technological growth is g ($A_t = (1+g)^t A_0$). Given a savings rate s, production function $F(K, L)$, and depreciation rate δ, the law of motion for capital K_t is given by

$$K_{t+1} = sF(K_t, A_t L_t) + (1-\delta)K_t.$$

(a) Define capital per efficiency units of labor by $\tilde{k}_t = K_t/(A_t L_t)$. Show that the law of motion for $\tilde{k}_t$ is given by

$$\tilde{k}_{t+1} = \frac{1}{(1+g)(1+n)}\left[sf(\tilde{k}_t) + (1-\delta)\tilde{k}_t\right]$$

where $f(k) = F(k,1)$. Be clear when you invoke an assumption about the production function F.

We start with the law of motion for capital K_t and perform the substitution $K_t = A_t L_t \tilde{k}_t$, yielding

$$A_{t+1}L_{t+1}\tilde{k}_{t+1} = sF(A_t L_t \tilde{k}_t, A_t L_t) + (1-\delta)A_t L_t \tilde{k}_t.$$

Next, we use that F is constant returns to scale to get $F(A_t L_t \tilde{k}_t, A_t L_t) = A_t L_t F(\tilde{k}_t, 1)$, which implies

$$A_{t+1}L_{t+1}\tilde{k}_{t+1} = sA_t L_t F(\tilde{k}_t, 1) + (1-\delta)A_t L_t \tilde{k}_t.$$

Factoring out $A_t L_t$ and dividing by $A_{t+1}L_{t+1}$ yields

$$\tilde{k}_{t+1} = \frac{A_t L_t}{A_{t+1}L_{t+1}} sF(\tilde{k}_t, 1) + (1-\delta)\tilde{k}_t.$$

Since $A_{t+1}/A_t = 1+g$ and $L_{t+1}/L_t = 1+n$, we thus arrive at

$$\tilde{k}_{t+1} = \frac{1}{(1+g)(1+n)}\left[sf(\tilde{k}_t) + (1-\delta)\tilde{k}_t\right]$$

where $f(k) = F(k,1)$.

(b) Show that in the long run (you may assume a balanced-growth path exists and that the model converges to it),

$$K_t/Y_t = s/(g + n + gn + \delta)$$

(where $Y_t = F(K_t, L_t)$ denotes output). Again, be clear when you invoke an assumption about the production function F. *Hint: solve for* $\tilde{k}_{ss}/f(\tilde{k}_{ss})$ *by evaluating the law of motion for* $\tilde{k}_t$ *at its steady state.*

Along a balanced growth path, we have

$$\begin{aligned}
\tilde{k}_{ss} &= \frac{1}{(1+g)(1+n)}\left[sf(\tilde{k}_{ss}) + (1-\delta)\tilde{k}_{ss}\right] \Leftrightarrow \\
(1+g)(1+n)\tilde{k}_{ss} &= sf(\tilde{k}_{ss}) + (1-\delta)\tilde{k}_{ss} \Leftrightarrow \\
\left[(1+g)(1+n) - (1-\delta)\right]\tilde{k}_{ss} &= sf(\tilde{k}_{ss}) \Leftrightarrow \\
\frac{\tilde{k}_{ss}}{f(\tilde{k}_{ss})} &= \frac{s}{g+n+gn+\delta}.
\end{aligned}$$

Note that $Y_t = F(K_t, A_t L_t) = A_t L_t F(\tilde{k}_t, 1) = A_t L_t f(\tilde{k}_t)$ (where the second inequality used that F is constant returns to scale and that $K_t = A_t L_t \tilde{k}_t$. We thus have

$$\frac{K_t}{Y_t} = \frac{\tilde{k}_t}{f(\tilde{k}_t)} = \frac{\tilde{k}_{ss}}{f(\tilde{k}_{ss})} = \frac{s}{g+n+gn+\delta}$$

along the balanced-growth path/ in the "long run".

(c) In the long run, what is the growth rate of GDP? What is the growth rate of GDP per capita? (simply state the answers)

The growth rate of GDP is $g+n+gn$ and the growth rate of GDP per capita is g.

(d) Assume $s = 0.15$ and $\delta = 0.05$. Assume GDP growth is 0.025. Compute the long-run capital-to-output ratio.

We plug in the numbers in the formula. The capital-output ratio is $0.15/(0.05+0.025) = 2$.

(e) Now assume GDP growth falls to 0. Compute the long-run capital-to-output ratio.

Again, we plug in the numbers in the formula. The capital-output ratio is $0.15/0.05 = 3$.

3 Romer model (40 points)

In this exercise, we go through the simple product-variety model with lab equipment. Recall that in this environment, the household's Euler equation, $r = \rho + \epsilon g$ holds, where ϵ is the inverse of the household's elasticity of intertemporal substitution. Final output at time t is produced under perfect competition, using labor and a range of intermediate inputs, indexed by i in the interval $[0, M_t]$ where M_t is the measure of product varieties available at time t. Final good production at time t is perfectly competitive and given by

$$Y_t = L^{1-\alpha} \int_0^{M_t} x_{ti}^{\alpha} di$$

where Y_t is output and each x_{ti} is the amount of intermediate product i used as input. Labor input is always equal to the fixed supply L. The coefficient α lies between zero and one. Each intermediate product is produced using the final good as input, one for one, and the patent for each intermediate product is owned by a monopolist.

(a) By computing the marginal product of x_{ti} in the final-goods production, show that the price of good x_{ti} (with the final good as numeraire) is $p_{ti} = \alpha L^{1-\alpha} x_{ti}^{\alpha-1}$.

Since the final good is produced under perfect competition, the price of good x_{ti} is $p_{ti} = \frac{d}{dx_{ti}} L^{1-\alpha} \int_0^{M_t} x_{ti}^{\alpha} di = \alpha L^{1-\alpha} x_{ti}^{\alpha-1}$

(b) The monopolist for good i sells its good at price p_{ti} and produces it at cost 1. Write down its maximization problem and show that the profit maximization implies

$$x = L\alpha^{2/(1-\alpha)},$$
$$\Pi = \frac{1-\alpha}{\alpha} L\alpha^{2/(1-\alpha)}.$$

The monopolist's problem is

$$\max_{x_{it}} p_{ti}(x_{ti}) x_{ti} - x_{ti} \Leftrightarrow \max_{x_{it}} \alpha L^{1-\alpha} x_{ti}^{\alpha} - x_{ti}$$

with first-order condition

$$\alpha^2 L^{1-\alpha} x_{ti}^{\alpha-1} = 1$$

or $x_{ti} = L\alpha^{2/(1-\alpha)}$.

Plugging this into the objective yields

$$\Pi = \alpha L^{1-\alpha} x^{\alpha} - x = \alpha L^{1-\alpha} L^{\alpha} \alpha^{2\alpha/(1-\alpha)} - L\alpha^{2/(1-\alpha)} =$$
$$= L\alpha^{(1+\alpha)/(1-\alpha)} - L\alpha^{2/(1-\alpha)} = L(1/\alpha - 1)\alpha^{2/(1-\alpha)} = \frac{1-\alpha}{\alpha} L\alpha^{2/(1-\alpha)}$$

where we note that $(1+\alpha)/(1-\alpha) + 1 = 2/(1-\alpha)$ explains the penultimate equality.

In particular, all intermediate-goods producer choose the same quantity and earn the same profit at all times.

(c) Gross domestic product is final-goods output net of intermediate-goods, $GDP_t = Y_t - \int_0^{M_t} x_{ti} di$. Show that GDP_t is proportional to M_t and thus that the growth rate of GDP equals $g = \frac{\dot{M}_t}{M_t}$.

We now know that $x_{it} = x$ so

$$Y_t = L^{1-\alpha} \int_0^{M_t} x_{it}^{\alpha} di = M_t L^{1-\alpha} x^{\alpha}$$

and thus

$$GDP_t = Y_t - \int_0^{M_t} x_{it} = Y_t - M_t x = M_t(L^{1-\alpha} x^{\alpha} - x),$$

i.e., GDP is proportional to M_t and the growth rate of GDP equals the growth rate of M_t, $g = \frac{\dot{M}_t}{M_t}$.

(d) Product variety grows at a rate that depends on the amount R_t of final output that is used in research. I.e., we have

$$\dot{M}_t = \lambda R_t$$

where R_t is the amount of final output that is used in research. Assume that the research sector is perfectly competitive with free entry, such that the flow of profit in the research sector must be zero. Each blueprint is worth Π/r to its inventor, the net present value of the profit stream from owning the patent to a new blueprint. Show that the "research-arbitrage equation" is

$$r = \lambda \Pi.$$

The gain from research is that new blueprints are discovered at rate λ, worth Π/r each. The gain per unit spent is thus $\lambda\Pi/r$. No arbitrage means that this gain should equal the cost of one unit of research, which is 1. Rearranging $\lambda\Pi/r = 1$, we arrive at

$$r = \lambda \Pi.$$

(e) Combining the research-arbitrage equation with the Euler equation, show that the endogenous growth rate is given by

$$g = \frac{\lambda \Pi - \rho}{\epsilon}.$$

We have two equations in two unknowns, the growth rate g and the interest rate r,

$$\begin{aligned} r &= \rho + \epsilon g, \\ r &= \lambda \Pi. \end{aligned}$$

Solving for g yields

$$g = \frac{\lambda \Pi - \rho}{\epsilon}.$$

(f) If the population size L doubles, does the growth rate g change and if so in which direction? Explain your answer.

Yes, the growth rate changes, it increases. We have from the previous part that $g = \frac{\lambda\Pi - \rho}{\epsilon}$. While λ, ρ, and ϵ are exogenous parameters, Π is an endogenous variable and equal to $\frac{1-\alpha}{\alpha} L \alpha^{2/(1-\alpha)}$ (part b). An increase in L thus increases Π, which increases g.