The Dixit-Stiglitz demand system, monpolistic competition and trade.

Economics students are generally well trained in perfectly competitive markets. Such markets are often thought to be characterized by well defined utility functions and homogenous production functions with constant and decreasing marginal productivities and constant returns to scale. Such models are very useful for understanding a wide set of economic mechanisms. Increasing returns, however, are known to be feature in many real world production processes. Often such production technologies are said to generate natural monopolies since an implication of increasing returns is that large production entities are more productive than smaller ones. Still, we know that many markets are characterized by many producers producing different varieties of the same products under increasing returns to scale technologies. A stylized description of such markets is the market type *monopolistic competition*. Such markets are characterized by many producers who each enjoy some market power but by free entry so that profit opportunities are limited.

Markets with monopolistic competition captures can easily be modeled with increasing returns to scale production technologies. This has shown very useful for many applications. Below we will consider international trade. We will demonstrate that such markets generate trade between similar countries and trade within the same industries, so called intra industry trade. These are known to be of great importance for real world trade flows. Since such trade can hardly be explained by comparative advantages, trade theory is fruitfully supplemented with monopolistic market approaches. However, monopolistic markets models have also had wide applications in many other economics topics, as e.g. growth theory, environmental economics, macroeconomics and microeconomics. The modeling framework presented below therefore have many applications.

The demand side

Monopolistic competitive markets must be characterized by a demand side that captures many product varieties. One specific approach for modeling this is to introduce a representative consumer who always demands the existing varieties. This can be interpreted literally so that one assumes that every consumer prefers variety in the consumption basket. Consequently the demand system we will describe is sometimes referred to as the “love of variety” approach. This may be misleading however since the underlying utility function implies *less* love of variety than standard utility functions of the Cobb-Douglas type (see below). Another interpretation is that the demand system is for a representative consumer who is an aggregate of many consumers with distinct individual preferences for each variety. Under some assumptions (which we will not discuss), it can be showed that such an aggregation may give rise to the preference structure for a representative consumer that we will introduce. The specific utility function introduced below is one with constant elasticity of substitution and it is referred to as a CES utility function.

The representative consumers’ preferences are described with the utility function:

$$U(q_1, q_2, ..q_N) = \left(\sum_{i=1}^{N} q_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

Above, q_i, denotes quantity of consumption good i and σ is the elasticity of substitution among varieties. Generally we will assume that σ>1.

If σ=∞, the utility function is just the sum of consumed quantities of each variety:

$$U = \left(\sum_{i=1}^{N} q_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} = \left(\sum_{i=1}^{N} q_i \right), \qquad \text{when } \sigma = \infty$$

The reason for this is that if σ=∞, both the exponents (within and outside the bracket) goes to one. Therefore the case when σ=∞ describes the case when the goods are perfect substitutes. In this case, consumers do not care if one good is substituted for an equal quantity of another.

If σ=1, the above utility function goes to infinity. A variety of the CES utility function were one imposes equal weights on all varieties converges to a Cobb-Douglas utility function for N goods when σ=1. We will rewrite the utility function with the use of the parameter θ=(σ-1)/σ. Now if σ=1, θ=0 (and if σ=∞, θ=1)

$$U = \left(\sum_{i=1}^{N} q_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} = \left(\sum_{i=1}^{N} q_i^{\theta} \right)^{\frac{1}{\theta}}$$

Taking the limit of the above when θ approaches 0 gives:

$$\lim_{\theta \to 0} U = \left(\sum_{i=1}^{N} q_i^{\theta} \right)^{\frac{1}{\theta}} = \infty$$

Now consider the function V below. In V, each element within the summation term are given equal weights.

$$V = \left(\sum_{i=1}^{N} \left(\frac{1}{N} \right) q_i^{\theta} \right)^{\frac{1}{\theta}}$$

Take the natural logarithm of the above expression to obtain:

$$\ln V = \ln\left(\sum_{i=1}^{N}\left(\frac{1}{N}\right)q_i^{\ \theta}\right)^{\frac{1}{\theta}} = \frac{\ln\left(\sum_{i=1}^{N}\left(\frac{1}{N}\right)q_i^{\ \theta}\right)}{\theta}$$

Letting θ approaching zero gives rise to an expression of the type (0/0). This can be handled with l'Hopital's rule which allows derivation of the nominator and the denominator with respect to θ. This produces the expression:

$$\lim_{\theta\to 0}\ln V = \lim_{\theta\to 0}\frac{\ln\left(\sum_{i=1}^{N}\left(\frac{1}{N}\right)q_i^{\ \theta}\right)}{\theta} = "\frac{0}{0}" = \lim_{\theta\to 0}\frac{\sum\left(\frac{1}{N}\right)q_i^{\theta}\ln q_i}{\sum\left(\frac{1}{N}\right)q_i^{\theta}} = \sum\frac{1}{N}\ln q_i$$

$$\rightarrow$$

$$\lim_{\theta\to 0}\ln V = \sum\frac{1}{N}\ln q_i$$

$$V = e^{\sum\frac{1}{N}\ln q_i}$$

The above expression is a Cobb-Douglas function for consumption of N goods where each has an expenditure share equal to (1/N). We have dropped the subscripts and superscripts from the summation sign above and will only reintroduce it when necessary henceforth. This proves that a CES function of the type V approaches a Cobb-Douglas function when θ goes to zero. For many maximization purposes, one uses the ratios of marginal utilities. These are the same for the two functions U and V.

The preference for variety can easily be seen from the utility function if we assume that all goods have the same price so that $p_i=p_j=p$, and are consumed in the same amount so that $q_i=q_j=q$. In this case, the utility function can be written:

$$U = \left(\sum_{i=1}^{N} q^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} = \left(Nq^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} = N^{\frac{\sigma}{\sigma-1}}q$$

Writing consumers income as Y, we know that Npq=Y and therefore that q=Y/Np. Substituting this into the above gives:

$$U = N^{\frac{\sigma}{\sigma-1}}q = N^{\frac{\sigma}{\sigma-1}}\left(\frac{Y}{Np}\right) = N^{\frac{\sigma}{\sigma-1}-1}\left(\frac{Y}{p}\right) = N^{\frac{1}{\sigma-1}}\left(\frac{Y}{p}\right)$$

Since σ>1, this implies that U increases with the available number of varieties. The higher is σ, the less does utility depends on the number of varieties. This is in line with intuition.

Now assume that incomes are generated from a mass of labour L that earns w per unit. This will generalize to total income in an economy populated with L workers. Therefore the budget constraint is wL. Consumers want to maximize their utility given this budget constraint. The corresponding Lagrange function is:

$$\mathbf{L} = \left(\sum q_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} - \lambda\left(\sum p_i q_i - wL \right)$$

First order conditions for utility maximization are:

$$\frac{\partial \mathbf{L}}{\partial q_i} = \frac{\sigma}{\sigma-1}\left(\sum q_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}-1} \frac{\sigma-1}{\sigma} q_i^{\frac{\sigma-1}{\sigma}-1} - \lambda p_i = 0$$

$$\rightarrow \left(\sum q_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{1}{\sigma-1}} q_i^{\frac{-1}{\sigma}} - \lambda p_i = 0$$

$$\rightarrow \left(\sum q_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{1}{\sigma-1}} q_j^{\frac{-1}{\sigma}} - \lambda p_j = 0$$

In the second line above the first order condition was somewhat simplified. In the third line the first order condition is repeated for another good, j. Maximization requires that similar conditions are fulfilled for all goods.

From the first order conditions we obtain:

$$\frac{q_i^{-\frac{1}{\sigma}}}{q_j^{-\frac{1}{\sigma}}} = \left(\frac{q_i}{q_j} \right)^{-\frac{1}{\sigma}} = \frac{p_i}{p_j}$$

$$\rightarrow \frac{q_i}{q_j} = \left(\frac{p_i}{p_j} \right)^{-\sigma}$$

$$q_i = q_j \left(\frac{p_i}{p_j} \right)^{-\sigma}$$

$$p_i q_i = q_j p_i \left(\frac{p_i}{p_j} \right)^{-\sigma} = q_j p_j^{\sigma} p_i^{1-\sigma}$$

$$\sum p_i q_i = \sum q_j p_i \left(\frac{p_i}{p_j} \right)^{-\sigma} = \sum_{i=1}^{N} q_j p_j^{\sigma} p_i^{1-\sigma} = q_j p_j^{\sigma} \sum p_i^{1-\sigma}$$

The second line is simply a reorganization of the first. In the third line we have solved for q_i. In the fourth line we multiply with p_i. In the fifth we sum over all q_i. The second equality reminds us that this summation is over i. Therefore elements in the sum with subscripts j can be multiplied outside the summation sign. From this we can solve for q_j:

$$q_j p_j^{\sigma} = \frac{\sum p_i q_i}{\sum p_i^{1-\sigma}}$$

$$q_j = \frac{p_j^{-\sigma} \sum p_i q_i}{\sum p_i^{1-\sigma}} = \frac{p_j^{-\sigma} wL}{\sum p_i^{1-\sigma}}$$

since

$$\sum p_i q_i = wL$$

The second line is the demand function for good j. It depends on the price, p_j, on total income, wL and on an expression that is a function of all prices in the economy. This is a function of a price index for all goods given by:

$$P = \left(\sum p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

Therefore the demand function can be written as:

$$q_j = \frac{p_j^{-\sigma} wL}{P^{1-\sigma}}$$

Note that elasticity of demand is just $-\sigma$:

$$\frac{dq_j}{dp_j}\frac{p_j}{q_j} = \frac{-\sigma p_j^{-\sigma-1} wL}{P^{1-\sigma}}\frac{p_j}{q_j} = \frac{-\sigma p_j^{-\sigma} wL}{P^{1-\sigma}}\frac{P^{1-\sigma}}{p_j^{-\sigma} wL} = -\sigma$$

The above implies that one term is neglected. That is the influence of firms's pricing decisions on the price index P. This is a valid approximation when there are many firms. When there are few firms however, the neglecting of each firms' influence on the price index is not valid. We will continue to ignore this term.

Why is P a price index for all prices? It can be shown that this is the cost for obtaining one unit of utility. Form the first order conditions we have:

$$\rightarrow \frac{q_i}{q_j} = \left(\frac{p_i}{p_j}\right)^{-\sigma}$$

$$q_i = q_j\left(\frac{p_i}{p_j}\right)^{-\sigma}$$

Insert this expression into the utility function:

$$U = \left(\sum_{i=1}^{N} q_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} = \left(\sum_{i=1}^{N} \left(q_j \left(\frac{p_i}{p_j} \right)^{-\sigma} \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} = \left(\sum_{i=1}^{N} \left(q_j^{\frac{\sigma-1}{\sigma}} p_j^{\sigma-1} p_i^{1-\sigma} \right) \right)^{\frac{\sigma}{\sigma-1}}$$

$$= \left(q_j^{\frac{\sigma-1}{\sigma}} p_j^{\sigma-1} \sum_{i=1}^{N} p_i^{1-\sigma} \right)^{\frac{\sigma}{\sigma-1}} = q_j p_j^{\sigma} \left(\sum_{i=1}^{N} p_i^{1-\sigma} \right)^{\frac{\sigma}{\sigma-1}}$$

Again we have set the elements containing subscript j outside the summation since summation runs over i. From the above, solve for q_j:

$$q_j = p_j^{-\sigma} \left(\sum_{i=1}^{N} p_i^{1-\sigma} \right)^{\frac{\sigma}{1-\sigma}} U$$

$$p_j q_j = p_j^{1-\sigma} \left(\sum_{i=1}^{N} p_i^{1-\sigma} \right)^{\frac{\sigma}{1-\sigma}} U$$

$$\sum_{j=1}^{N} p_j q_j = \sum_{j=1}^{N} p_j^{1-\sigma} \left(\sum_{i=1}^{N} p_i^{1-\sigma} \right)^{\frac{\sigma}{1-\sigma}} U = \left(\sum_{i=1}^{N} p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}} U$$

$$E = \left(\sum_{i=1}^{N} p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}} = P$$

In the second equation we multiplied the solution for q_j with p_j. In the third line we summed over all js. Since the sum over all js and over all is involves summing over all prices the sums can be expressed as in the last equality in third line. In the second line we set U=1 and write the corresponding sum as E. These are the expenditures for one unit of utility and therefore the price index for all goods.

The production side

It is important the demand functions that we derived above has the price elasticity $-\sigma$. Since $\sigma>1$ this implies that firms facing these demand functions have some market power. It is well known that when there are increasing returns in production, marginal costs are lower than average costs. Therefore perfectly competitively pricing where price equals marginal costs cannot sustain production when there are increasing returns to production. We will allow firms to price monopolistically in our model. Therefore prices are set above marginal costs and there are operating surpluses. This allows increasing returns since the operating surplus can cover also fixed costs in production. The

monopolistic competition market form also assumes that there is free entry to the market. Therefore profits are assumed to be squeezed away.

Firms are assumed to produce one variety each. Since there are increasing returns to scale, production costs are lower when they concentrate on one product variety only. Since producers of single varieties have market power it does not make sense to produce the same variety as other firms. Thus firms produce distinct varieties.

We will assume that labour is the only factor of production. Production takes place under increasing returns to scale. In order to operate firms have to use the fixed amount of labour f. The cost of this, wf, is therefore a fixed cost component. In addition production also requires b units per product. The use of labour in production of variety j is therefore given in the first equation below. Total costs are given in the second equation.

$$l_j = f + bq_j$$
$$C_j = w\left(f + bq_j\right)$$

Note that firms are assumed to have similar cost structures so that w, b and f are similar across firms.

Profits are therefore:

$$\pi_j = p_j q_j - w\left(bq_j + f\right)$$

Remember that demand is given as

$$q_j = \frac{p_j^{-\sigma} wL}{P^{1-\sigma}}$$

Profit maximization therefore requires setting the derivative of the profit function with respect to q_j equal to zero:

$$\frac{d\pi_j}{dq_j} = p_j + \frac{dp_j}{dq_j} q_j - wb = 0$$

$$\rightarrow$$

$$p_j \left(1 + \frac{dp_j}{dq_j} \frac{q_j}{p_j} \right) = wb$$

$$p_j \left(1 - \frac{1}{\sigma} \right) = p_j \left(\frac{\sigma - 1}{\sigma} \right) = wb$$

$$p_j = \frac{\sigma}{\sigma - 1} wb$$

$$\pi_j = p_j q_j - w(bq_j + f) = \frac{\sigma}{\sigma - 1} wbq_j - w(bq_j + f) = w\left(\frac{\sigma - (\sigma - 1)}{\sigma - 1} bq_j - f \right) = w\left(\frac{1}{\sigma - 1} bq_j - f \right)$$

Above we also solved for the optimal price in the fourth line. Note that prices are set as a markup over marginal costs ($\sigma/(\sigma-1)>1$). In the fifth line we inserted the resulting price into the profit equation and found the resulting expression for profits. Note that prices are equal for all firms.

The assumption of free entry implies that profit opportunities are eliminated. We can solve for each firm's produced quantity by setting profits equal to zero:

$$\pi_j = w\left(\frac{1}{\sigma - 1} bq_j - f \right) = 0$$

$$\rightarrow$$

$$\frac{1}{\sigma - 1} bq_j = f$$

$$q_j = \frac{\sigma - 1}{b} f$$

The above equation depends only on constants that are common for all firms. We have therefore established that all firms produce the same quantities.

We have determined both common prices and common produced quantities. It remains to determine the number of firms and thus the number of product varieties available in the market. This can be done using the labour market equilibrium:

$$Nl = L$$
$$N(bq + f) = L$$
$$N\left(b\frac{\sigma - 1}{b}f + f\right) = N(\sigma f - f + f) = N\sigma f = L$$
$$N = \frac{L}{\sigma f}$$

Welfare

The above model can be interpreted as a general equilibrium model for a closed economy. Consumers demand consumption goods according to their demand functions. The demand functions are derived from the underlying utility function. Consumers earn their income from being part of the labour force. Firms employ workers for production of their individual varieties, produced under increasing returns to scale. The firms price monopolistically but free entry ensures that profits are squeezed away. Since firms are symmetrical, their price and produced quantities are similar. The number of firms is determined by the available labour force.

We noted when we introduced the utility function that when $p_i=p_j=p$ and $q_i=q_j=q$ the utility function can be written:

$$U = \left(\sum_{i=1}^{N} q^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} = \left(Nq^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} = N^{\frac{\sigma}{\sigma-1}}q$$

Consumers income is wL, and we know that Npq=wL and therefore that q=wL/Np. Substituting this into the above gives:

$$U = N^{\frac{\sigma}{\sigma-1}}q = N^{\frac{\sigma}{\sigma-1}}\left(\frac{wL}{Np}\right) = N^{\frac{\sigma}{\sigma-1}-1}\left(\frac{wL}{p}\right) = N^{\frac{1}{\sigma-1}}\left(\frac{wL}{p}\right)$$

Inserting for the number of firms and prices give:

$$U = N^{\frac{1}{\sigma-1}}\left(\frac{wL}{p}\right) = N^{\frac{1}{\sigma-1}}\left(\frac{wL}{p}\right) = \left(\frac{L}{\sigma f}\right)^{\frac{1}{\sigma-1}}\left(\frac{wL}{\frac{\sigma}{\sigma-1}wb}\right) =$$
$$L^{\frac{\sigma}{\sigma-1}}\sigma^{\frac{\sigma}{1-\sigma}}f^{\frac{1}{1-\sigma}}\frac{(\sigma-1)}{b}$$

Since σ>1, this implies that U increases with the available labour force. The higher is σ, the less does utility depends on the labour force. In this model, increased population implies that a larger number of product varieties is on the market. Each firm produces the same amount. This is a consequence of the specific utility function that is used. In

Krugman (1979) and in Feenstra (2004) a similar model to the one above is used, but with a more general utility function. In that case, the elasticity of substitution depends on the consumption of each good. Therefore also the demand elasticity depends on substation. When demand elasticity increases with consumed quantity, increased population results both in more varieties and increased production of each variety and therefore lower costs (because of increasing returns) and lower prices.

Trade

The model sketched above was for one economy only. One can analyze trade within this model by assuming two countries described as above. Transition from trade to autarky within that model involves expanding the size of the two economies. From the welfare considerations above it seems clear that this increases welfare. More product varieties become available. For a more general model, the effects of trade may potentially be several:

- Trade increases market share and competition. This could imply both increased production per firm so that average costs decrease and reduced prices.
- Trade increases the number of firms so that more product varieties become available. With the CES utility function we have used, effects of trade is limited to this mechanism. In the expressions for firms' output, produced quantity does not depend on market size. In this model therefore, firms' production is the same with and without trade, and trade merely involves more products being available.

Also, the model we have described is a one-sector model. The product varieties are imperfect substitutes but are of the same types. Consumers consume some of all varieties available. Trade between two countries is therefore intra-industry trade, i.e. trade within the same industries.
Suppose that there are two countries characterized as in the model above. Assume that their technologies and their preferences are identical. The two countries are only allowed to differ in size which are L_H and L_F, respectively. Since firms and prices are identical, we drop the firm subscripts from now. Subscripts will instead be used to indicate counties. Since prices are the same also wages are the same, so that $w=w_H=w_F$.

Prices, quantities and the number of firms are given by the equations:

$$p = \frac{\sigma}{\sigma - 1} wb$$

$$q = \frac{\sigma - 1}{b} f$$

$$N_i = \frac{L_i}{\sigma f} \qquad i = H, F$$

As we have noted, consumers consume some of all varieties available, also of varieties produced abroad. Thus exports from country i to country n is:

$$X_{in} = N_i p_{in} q_{in} = N_i p_i \frac{p_{in}^{-\sigma} w L_n}{P_n^{1-\sigma}}$$

$$q_{in} = \frac{p_{in}^{-\sigma} w L_n}{P_n^{1-\sigma}}$$

The first line above gives exports as the value of sales per firm from country i to country n, ($p_{in}q_{in}$), times the number of firms in country i. In the second equation in the first line we have inserted the demand function. This is given in the second line. Demand for a good produced in country i in country n is given by the same demand function as derived above, but with due notification that income and the price index is for country n. The price index in country n is given by:

$$P_n = \left(N_i p_{in}^{1-\sigma} + N_n p_{nn}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

The fraction of total income in country n spent on goods from country is

$$\lambda_{in} = \frac{X_{in}}{X_n} = \frac{N_i p_{in} q_{in}}{N_i p_{in} q_{in} + N_n p_{nn} q_{nn}} = \frac{L_i}{L_i + L_n}$$

The first equation simply defines income in country n as X_n in the denominator. Since all income is spent it is the sum of what is spent in country i and what is spent in country n. This is the denominator in the second equation. The nominator simply uses the definition of exports from above. In the third equation we have used the fact that prices and per firm quantities are the same in the two countries. These can therefore be cancelled from the equation. Furthermore, the number of firms in both countries is proportional to the two countries. The result follows. From the above we also obtain total exports from country i to country n:

$$X_{in} = \frac{L_i}{L_i + L_n} w L_n$$

This follows since total income in country n is wage income. It follows that trade is balanced (you can try to show this). You should also note that exports from i to n depends on the two countries, sizes.

With trade we also find welfare by using the expression from above:

$$U_i = \left(L_H + L_F\right)^{\frac{\sigma}{\sigma-1}} \sigma^{\frac{\sigma}{1-\sigma}} f^{\frac{1}{1-\sigma}} \frac{(\sigma - 1)}{b}$$

Gains from trade come from increased product variety. Also note that the welfare increase is largest for the smallest country.

More than one industry

Sometimes the Dixit Stiglitz demand system is used in combination with demand for other goods. This is particularly convenient in models for more than one industry. One might, for instance, assume that there is one standard industry producing a homogenous good under constant returns to scale and one sector producing varieties of a differentiated good under increasing returns to scale. A common example where this approach is used is for models with an agriculture sector producing a homogenous good under constant returns to scale and a manufacturing sector producing differentiated goods under increasing returns to scale.

Below we will formulate this as a two stage optimization problem. The first stage is to choose between consumption of the homogenous good and an aggregate of the differentiated good. This first stage therefore determines the expenditures on the homogenous good and the aggregate of the differentiated good, respectively. In the second stage the quantities of each variety of the differentiated good, given the expenditures on these goods, are determined.

Consumers' utility function is written

$$U = C_a^{1-\alpha} C_m^{\alpha}$$

$$C_m = \left(\sum_{i=1}^{N} q_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

The first equation is the overall utility function for consumption of C_a and the aggregate of differentiated good C_m. The utility function is a standard Cobb-Douglas utility function. The second equation is a CES function of the type we introduced above. This CES function has the same characteristics as we have already described.

We will use the C_a good as numeraire and set its price level equal to 1. Prices on the differentiated good are described with the price index above; written P. Consumers income are their wage income. We therefore have the budget constraint:

$$C_a + PC_m = wL$$

The above constraint simply expresses that total expenditures on C_a and C_m equal income. The optimization problem is constrained by the budget constrain and can therefore be solved with use of the Lagrange method. The Lagrangian is:

$$\mathbf{L} = C_a^{1-\alpha} C_m^{\alpha} - \lambda\left(C_a + PC_m - wL\right)$$

First order conditions are:

$$\frac{d\mathbf{L}}{dC_a} = (1-\alpha)C_a^{-\alpha}C_m^{\alpha} - \lambda = 0$$
$$\frac{d\mathbf{L}}{dCm} = \alpha C_a^{1-\alpha}C_m^{\alpha-1} - \lambda P = 0$$
$$\rightarrow$$
$$\frac{\alpha}{1-\alpha}C_m^{-1}C^{\alpha} = P$$
$$\alpha C_a = (1-\alpha)C_m P$$
$$C_a = \frac{(1-\alpha)}{\alpha}C_m P$$

From the first order conditions we obtained the three last equations above. The last equation gives consumption of C_a as a function of prices and the aggregate consumption C_m. From the budget equation we obtain:

$$\frac{(1-\alpha)}{\alpha}PC_m + PC_m = wL$$
$$PC_m = \alpha wL$$
$$C_a = (1-\alpha)wL$$

The above demonstrates that α and (1-α) are the expenditures for C_m and C_a respectively. This was the first stage budgeting. The second stage involves choosing each q_i given aggregate expenditures on the aggregate good C_m. We have established that aggregate expenditures on C_m is αwL. The corresponding Lagrangian for this constrained maximization problem is:

$$\mathbf{L}^* = \left(\sum q_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} - \lambda\left(\sum p_i q_i - \alpha wL\right)$$

The * simply indicates that the above Lagrangian is different from the one above. The first order conditions for optimisation good i and good j are:

$$\frac{\partial \mathbf{L}^*}{\partial q_i} = \frac{\sigma}{\sigma-1}\left(\sum q_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}-1}\frac{\sigma-1}{\sigma}q_i^{\frac{\sigma-1}{\sigma}-1} - \lambda p_i = 0$$
$$\rightarrow \left(\sum q_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} q_i^{\frac{-1}{\sigma}} - \lambda p_i = 0$$
$$\rightarrow \left(\sum q_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} q_j^{\frac{-1}{\sigma}} - \lambda p_j = 0$$

It is obvious that the structure of this problem is similar to that of maximizing U subject to the budget constraint we described in the one sector case. Also in this case we obtain:

$$\frac{q_i^{-\frac{1}{\sigma}}}{q_j^{-\frac{1}{\sigma}}} = \left(\frac{q_i}{q_j}\right)^{-\frac{1}{\sigma}} = \frac{p_i}{p_j}$$

$$\rightarrow \frac{q_i}{q_j} = \left(\frac{p_i}{p_j}\right)^{-\sigma}$$

$$q_i = q_j\left(\frac{p_i}{p_j}\right)^{-\sigma}$$

The last equation above gives demand for good i as a function of demand for good j and their relative price. Insert this expression into the sub utility function C_m to obtain:

$$C_m = \left(\sum_{i=1}^{N} q_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} = \left(\sum_{i=1}^{N}\left(q_j\left(\frac{p_i}{p_j}\right)^{-\sigma}\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} = \left(\sum_{i=1}^{N} q_j^{\frac{\sigma-1}{\sigma}} p_j^{\sigma-1} p_i^{1-\sigma}\right)^{\frac{\sigma}{\sigma-1}} =$$

$$\left(q_j^{\frac{\sigma-1}{\sigma}} p_j^{\sigma-1}\sum_{i=1}^{N} p_i^{1-\sigma}\right)^{\frac{\sigma}{\sigma-1}} = q_j p_j^{\sigma}\left(\sum_{i=1}^{N} p_i^{1-\sigma}\right)^{\frac{\sigma}{\sigma-1}}$$

In the second equation the expression was inserted into the utility function. In the third equation the expression was simplified somewhat. In the fourth equation we used the fact that summation runs over i and not over j. The resulting equation can be solved for the demand for good j:

$$q_j = \frac{p_j^{-\sigma} C_m}{\left(\sum q_i^{1-\sigma}\right)^{\frac{\sigma}{\sigma-1}}}$$

Note that total expenditures on the C_m goods are E:

$$E = \sum_{j=1}^{N} p_j q_j = \sum_{j=1}^{N} \frac{p_j^{1-\sigma} C_m}{\left(\sum_{i=1}^{N} p_i^{1-\sigma}\right)^{\frac{\sigma}{\sigma-1}}} = C_m \sum_{j=1}^{N} p_j^{1-\sigma} \left(\sum_{i=1}^{N} p_i^{1-\sigma}\right)^{\frac{-\sigma}{\sigma-1}} =$$

$$C_m \left(\sum_{i=1}^{N} p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

Above, the last equation is valid since summing over all j's and all i's involves summing over all goods and therefore over the same goods. Note that the last equation contain the same expression as we have defined as the price index before, P:

$$P = \left(\sum_{i=1}^{N} p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

From the first stage budgeting we also have the results that

$$C_m = \alpha \frac{wL}{P}$$

The demand function can therefore be written:

$$q_j = \frac{p_j^{-\sigma} C_m}{\left(\sum q_i^{1-\sigma}\right)^{\frac{\sigma}{\sigma-1}}} = \frac{p_j^{-\sigma} \alpha wL}{\left(\sum q_i^{1-\sigma}\right)^{\frac{\sigma}{\sigma-1}} P} = \frac{p_j^{-\sigma} \alpha wL}{\left(\sum q_i^{1-\sigma}\right)^{\frac{\sigma}{\sigma-1}} \left(\sum q_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}}} =$$

$$\frac{p_j^{-\sigma} \alpha wL}{\left(\sum q_i^{1-\sigma}\right)} = \frac{p_j^{-\sigma} \alpha wL}{P^{1-\sigma}}$$

The above is the same demand function that we arrived at in the one sector case except for the fact that only a fraction, α, of total income is used for the differentiated goods C_m.