

Economic Dynamics and Uncertainty

Final Exam – Fall 2024

Each roman numbered subquestion counts equally to the exam. If you think there is a mistake or ambiguous statement, briefly explain the issue, state how you interpret or fix the potential issue, and continue based on your assumption.

1. The Euler equation in different forms (60%). A social planner with an infinite planning horizon and utility discount factor β maximizes a representative agent's utility $U(c_t)$. Economic production accumulates the stock of goods x_t . Some part of it is consumed, the other part is invested and produces more goods according to the production function f so that

$$x_{t+1} = f(x_t - c_t).$$

- i) Spell out the social planner's Bellman equation including all necessary substitutions and/or constraints required to solve the problem.

Solution: The Bellman or dynamic programming equation is

$$v(x_t) = \max_{c_t} [U(c_t) + \beta v(f(x_t - c_t))] \quad (1)$$

with x_0 given and $c_t, x_t \geq 0$.

- ii) Use the envelope theorem and the first order condition to derive the Euler equation from the dynamic programming equation.

Solution: Step 1: First order condition of r.h.s. Bellman

$$U'(c_t) = \beta v'(x_{t+1}) f'(x_t - c_t) \quad (2)$$

or

$$v'(x_{t+1}) = \frac{U'(c_t)}{\beta f'(x_t - c_t)} \quad (3)$$

Step 2: Derive Bellman equation w.r.t. x_t and make use of the Envelope theorem

$$v'(x_t) = \beta v'(x_{t+1}) f'(x_t - c_t).$$

It's easiest to advance the equation by one period

$$v'(x_{t+1}) = \beta v'(x_{t+2}) f'(x_{t+1} - c_{t+1}). \quad (4)$$

Step 3: We can use equation (3) in t and $t + 1$ to substitute the unknown value function in the previous equation

$$\frac{U'(c_t)}{\beta f'(x_t - c_t)} = \beta \frac{U'(c_{t+1})}{\beta f'(x_{t+1} - c_{t+1})} f'(x_{t+1} - c_{t+1}) = U'(c_{t+1}) \quad (5)$$

We successfully eliminated the unknown value function and rearranging yields

$$U'(c_t) = \beta f'(x_t - c_t) U'(c_{t+1}) \quad (6)$$

which is the Euler equation.

iii) How do Bellman equation and Euler equation change if productivity is uncertain, say

$$x_{t+1} = \tilde{\epsilon}_t f(x_t - c_t)$$

where $\tilde{\epsilon}_t$ is drawn from a sequence of independent and identically distributed shocks? Please re-derive the Euler equation, careful noting that some of the transformations might no longer work when faced with expectations. Please mark all (one-step-ahead) stochastic variables with a tilde ($\sim$). The result will be most likely as expected, so if you cannot derive the result, you are still welcome to write down your guess for the Euler equation, stating explicitly that it is but a guess.

Solution: The Bellman or dynamic programming equation becomes

$$U'(c_t) = \beta \mathbb{E}_t \tilde{\epsilon} f'(x_t - c_t) U'(c_{t+1}) \quad (7)$$

Step 1: First order condition of r.h.s. Bellman

$$U'(c_t) = \beta \mathbb{E}_t \tilde{v}'(x_{t+1}) \tilde{\epsilon}_t f'(x_t - c_t). \quad (8)$$

Step 2: Derive Bellman equation w.r.t. x_t and make use of the Envelope theorem

$$v'(x_t) = \beta \mathbb{E}_t \tilde{v}'(x_{t+1}) \tilde{\epsilon}_t f'(x_t - c_t). \quad (9)$$

Now we have to observe that, based on comparing equations 8 and (9), $v'(x_t) = U'(c_t)$. Given that, conditional on the the state variable, the problem is the same in the subsequent period, we also know that $v'(x_{t+1}) = U'(c_{t+1})$. Alternatively, you can advance envelope condition by one period to see that

$$v'(x_{t+1}) = \beta \mathbb{E}_{t+1} v'(x_{t+2}) \tilde{\epsilon}_{t+1} f'(x_{t+1} - c_{t+1}). \quad (10)$$

gives the same r.h.s. as when advancing the first order condition by one period.

Step 3: Using this insight in equation (8), we find

$$U'(c_t) = \beta \mathbb{E}_t \tilde{U}'(x_{t+1}) \tilde{\epsilon}_t f'(x_t - c_t). \quad (11)$$

which is the Euler equation under uncertainty.

- iv) Bring your Euler equation to the form $\mathbb{E} \tilde{M} \tilde{R} = 1$. What does $\tilde{M}$ stand for in general? What is its form in this particular model? What is $\tilde{R}$ in this model? Interpret the Euler equation.

If you believe that your Euler equation is wrong or you could not derive any Euler equation, go ahead and just interpret the general equation $\mathbb{E} \tilde{M} \tilde{R} = 1$ as we did in class.

Solution:

$$\tilde{M} = \frac{\beta U'(c_{t+1})}{U'(c_t)} \quad \text{and} \quad \tilde{R} = f'(x_t - c_t) \quad (12)$$

In general, $\tilde{M}$ is the stochastic discount factor, which here is the ratio of discounted future and present marginal utility. $\tilde{R}$ is the return of the economy given by marginal productivity at the investment level.

The Euler equation states that, in expectation, the marginal rate of substitution between consumption today and tomorrow has to equal economic productivity or the rate of technical transformation.

- v) If we assume that $\tilde{M}$ is linear in some observable factor we have what is known as a factor pricing model. Let us assume, as we did in this model, that a representative agent maximizes utility. State two alternative assumptions that allow us to transform the Euler equation of the form $\mathbb{E} \tilde{M} \tilde{R} = 1$ into a factor pricing model with the pricing factor consumption (or consumption growth). Briefly explain what this factor model says about the excess return of a given asset. You are welcome to write down a corresponding equation.

Solution: We obtain the consumption based asset pricing model by assuming, e.g.,

- quadratic utility or
- power utility in combination with normally distributed consumption growth and investment returns.

We expect a discussion mentioning that the model relates the asset's excess return to the assets covariance with consumption (or consumption growth). The states equation and corresponding discussion could be based e.g. on variations of this formula

$$\mathbb{E} \tilde{R}_i - R_f = \beta(\tilde{R}_i, \tilde{R}_x) \eta$$

or variations and briefly explain β as the regression coefficient and η as the risk premium.

- vi) Briefly discuss one alternative factor pricing model that differs from both, the consumption-based factor pricing model discussed above and the classic CAPM factor pricing model. Please motivate this alternative model (not the CAPM!) and explain what differs w.r.t. the consumption-based approach.

Solution: Two examples we discussed in class are

- the Fama French model with multiple pricing factors. One way of motivating this model is by assuming a time-changing(linear) relationship between a single (conditional) pricing factor and the stochastic discount factor can lead to multiple unconditional pricing factors.
- the Epstein-Zin-Weil preference model, which gives rise to the power of the market portfolio as an alternative (IES=1) or additional pricing factor (either answer here is good), which is the power of the market portfolio return. It's motivation is the distinction between individuals' desire to smooth consumption over time and their risk aversion.

2. Using the Bellman equation to solve a dynamic economic problem (40%).

This problem discusses Bellman's dynamic programming approach to solving economic optimization problems with many periods. When asked to contrast it with alternative methods, we have in mind classical optimization problems that take the sum of utility over time as the point of departure.

- i) What is the main advantage of using the Bellman equation as compared to other methods when solving a dynamic economic problem with many periods numerically?

Solution: Solving a many-period problem directly, requires optimizing over a decision-tree that quickly branches out. We would have to solve for a large number of paths and conditional control variables. The Bellman equation breaks up the many-period problem into a sequence of two period problems. In every step, we only have to take care of the uncertainty from one period to the next (no branching out).

- ii) How does the main challenge of solving the Bellman equation differ from a "classical optimization approach"?

Solution: The challenge in solving the Bellman equation is finding the value function. The (or each individual) optimization problem is generally quite simple conditional on knowing the value function. However, initially we do not know the value function and, thus, the Bellman equation is a functional equation.

- iii) Assume that someone write's down a dynamic economic problem with continuous states and gives you a trial solution for the value function. State verbally the main steps you would take to verify (or falsify) that this value function indeed solves the Bellman equation.

Solution:

1. Write down the Bellman equation, inserting the trial solution (= functional form) for the value function on both the left and the right side of the Bellman equation. This trial solution will generally contain some free parameters that are to be determined in the solution process.
2. Solve the first order conditions of the optimization problem on the right hand side of the Bellman equation
3. Plug the solutions for the control variables back into the Bellman equation
4. Check that the resulting equation can indeed be satisfied by choosing possible free parameters of the value function appropriately. This step generally requires to sort the Bellman equation in terms of terms depending on the different states (or combinations, or powers of states).
5. If the euqation can be satisfied for all reasonable values of the state variabels, then the value function indeed solves the problem.

- iv) Assume that the trial solution did not work out and that you have to solve the problem numerically. Why can't we simply solve the right side of the Bellman equation for all possible states of the dynamic system? Briefly discuss two different approaches to approximate the solution of the Bellman equation.
- You do not have to discuss the full solution approach, we are merely interested in a single iteration (or update of the value function) and how you suggest to deal with the issue that we cannot solve the right side of the Bellman equation for all possible states.

Solution: With a continuous state space we have to evaluate the r.h.s. of the Bellman function for uncountably many realizations of the state(s), and store the result. But no computer as (uncountably) infinite memory. Thus, we discussed two options:

1. discretize the state (and control) space. Solve the r.h.s. Bellman equation only on the discretized grid.
2. pick a (finite subset of) basis functions to approximate the value function over the continuous state space and evaluate the r.h.s. of the Bellman equation at a finite number of grid points. We use these evaluation points to then update/pin down the basis coefficients.

See lecture slides for details.