1. Public goods and environmental economics:
Requires answers:
Definition of public goods: Non-rivalry in consumption, non-excludability of consumers.
Give examples within environmental economics: clean air, existence of eco-systems, amenity variables, etc.
Deriving demand for an environmental good, $E$, as a public good and $x_i$ vector of private goods

$$U_i = U_i(x_i, E), U_{iE} > 0, U_{iE}^* < 0$$

First step express marginal utility for the environmental good in money.
Second step identify marginal willingness to pay in money for the environmental good, $U_{iE}'(x_i, E)$, as the demand function.
Third step stress that since $E$ is the same argument for all consumers the total marginal willingness to pay in money, i.e. the total demand function, is found by vertical summation

2. It is smart to see the problem of explaining $B(P)$ and $c(P)$ using the same underlying production model.
The benefit function $B(P)$, $B$ benefit in money, $P$ pollution.
The benefit function is used to characterise the situation of the polluter. The function measures the increase in profits by increasing the pollution, but the function is only valid up to a certain amount of pollution. If we start at this amount of pollution and decrease pollution the benefit function measures the cost of reducing pollution.

The polluter may in general reduce pollution by producing less, substitute between inputs, and purify by end-of-pipe means. It is reasonable to assume $B'(P) > 0$, $B''(P) < 0$.

Purification cost function $c = c(P)$, $c'<0$, $c''>0$:
Purification implies input substitution and end-of-pipe purification activities for constant output.

To understand both functions it is necessary to model the underlying production possibilities showing effects of output reduction, substitution and end-of-pipe purification activities explicitly. This is most conveniently done using the Frisch concept of factorially determined multi-output production:

$$y = f(x_1, x_2), f'_1 > 0, f'_2 = 0$$
$$P = h(x_1, x_2), h'_1 > 0, h'_2 < 0$$

Here $x_1$ is a vector of ordinary inputs (capital, labour, energy, materials, KLEM), and $x_2$ is a vector of purification inputs. To show substitution effects at least two production inputs have to be specified. We simplify and drop to show substitution by only considering one ordinary input.

Maximising profit to given output price $p$ and input prices $q_1, q_2$ with a restriction on pollution

$$\text{Max } \pi = py - q_1 x_1 - q_2 x_2$$
$$\text{s.t.}$$
$$y = f(x_1, x_2), P = h(x_1, x_2)$$
$$h(x_1, x_2) \leq P^R$$

Inserting the production function the Lagrangian is
The first-order conditions are
\[
\frac{\partial L}{\partial x_1} = p f_1' - q_1 - \lambda h_1' \leq 0 \\
\frac{\partial L}{\partial x_2} = p f_2' - q_2 - \lambda h_2' \leq 0
\]

Inequality in the first condition implies that zero amount of $x_1$ should be used, but then there is no production and of course no purification input, i.e. end of story. We will therefore disregard this possibility and assume that the first condition always hold with equality. If the second condition holds with inequality it means that no purification will take place. This may be an interesting possibility, but will typically only happen if the pollution constraint is not reached, i.e. the constraint is not binding.

Let us assume that this happens. This means that $P^R$ is set so high that when profit is maximised the resulting pollution is less. Then the shadow price $\lambda$ is zero. Furthermore, no purification inputs will be used, $x_2 = 0$. So the pollution variable can be ignored. In this situation we have the first-order condition
\[
p f_1'(x_1^*, 0) = q_1, \ x_2 = 0 \Rightarrow P^z = h(x_1^*, 0) \text{ and } P^z < P^R,
\]
where $x_1^*$ is the profit-maximising level of the ordinary input. The solution for pollution defines the limit $P^z$ for the amount of pollution up to which benefit for the firm is positive.

When we have $P^z > P^R$ the pollution constraint will become active with typically a positive shadow price $\lambda$. All three (here only two) possibilities of keeping pollution at the required constraint will be used; input substitution, purification input positive, and output reduction. This is abatement.

The endogenous variables are functions of the exogenous variables and the profit function can be written:
\[
\pi^* = pf(x_1(p, q_1, q_2, P^R), x_2(p, q_1, q_2, P^R)) - q_1 x_1(p, q_1, q_2, P^R) - q_2 x_2(p, q_1, q_2, P^R) = \pi(p, q_1, q_2, P^R) = B(P), 0 \leq P \leq P^z
\]
The prices are assumed to be constant.
The purification cost function is found by specifying a given output level, and then minimizing costs for a restricted amount of pollution \(0 < P < P^\pi\), specifying only one ordinary input and one purification input:

\[
\text{Min} \sum_{i=1}^{2} q_i x_i
\]

\[s.t.
\]
\[y = f(x_1, x_2) \geq y^o
\]
\[P = h(x_1, x_2) \leq P^e
\]

The Lagrangian is:

\[
L = - \sum_{i=1}^{2} q_i x_i
\]

\[-\gamma (f(x_1, x_2) + y^o)
\]

\[-\lambda (h(x_1, x_2) - P^e)
\]

The first-order conditions are

\[
\frac{\partial L}{\partial x_1} = -q_1 + \gamma f_1' - \lambda h_1' \leq 0
\]

\[
\frac{\partial L}{\partial x_2} = -q_2 + \gamma f_2' - \lambda h_2' \leq 0
\]

Assuming interior solution we have 4 equations to solve for \(x_1, x_2, \lambda, \gamma\) as functions of exogenous variables. Inserting in the factor outlay expression yields the cost function

\[
c(y^o, q_1, q_2, P^e), c_{y^o}' > 0, c_{q_1}' > 0, c_{P^e}' < 0
\]

Dropping the first three variables as arguments (including them in the functional form because they are assumed to be constant) yields the purification cost function \(c(P)\). The derivative of the amount of pollution is only negative up to the limit \(P^e\). To see substitution at work at least two ordinary inputs must be specified.

Abatement means using all options to reduce pollution, including reduced production, while purification means using all options, but keeping production
fixed. If output is kept fixed using the benefit function this implies that \( B(P) \) is identical to \( c(P) \), but with opposite interpretation of the sign of \( B \) and \( c \).

3. The Coase theorem using the basic model:
   First the Coase theorem has to be stated: If bargaining can take place without transaction costs, full information, price-takers, no wealth effect, and with unique property rights to the environmental medium in question, then bargaining between profit-maximising (utility-maximising) parties will lead to the socially optimal solution independent of which party is given the property right.
   The demonstration can be based on the figure from Lecture 3:
Problems with the Coase theorem:
There are problems with the realism of the underlying assumptions, no transaction costs, perfect information, etc.

4. Pigou tax using \(c(P)\) and \(D(P)\):

Social solution

\[
\text{Min} \{c(P) + D(P)\}
\]

First-order condition

\[
c'(P) + D'(P) = 0
\]
Economic implications:
Tax solution gives reduced profit to the firm, may lead to too many closures since tax amount is greater than total social damage.
Efficiency the same in the short run, but tax may give a dynamic stimulation to technical change.

5. Spatial model:
The transport coefficient has to be introduced. Damage from pollution from $N$ different sources

$$D = \bar{D}(P_1, \ldots, P_N) = D(a_1P_1, \ldots, a_NP_N)$$

Assumption: transport coefficient (the amount of one unit of emission from a source $i$) for emission from a source $i$ is not influenced by the amount discharged

The Pigou tax: The social problem

$$\text{Min} \left\{ D(\sum_{i=1}^{N} a_iP_i) + \sum_{i=1}^{N} c_i(P_i) \right\}$$

First-order condition:
\[ D' \alpha_i + c'_i(P_t) = 0, \ i = 1, \ldots, N \]

The firm problem:

\[ \text{Min}\{c_i(P_t) + tP_t\} \]

First-order condition:

\[ c'_i(P_t) + t = 0 \]

Setting the socially optimal tax rate:

\[ t = -c'_i(P_t) = D' \alpha_i \Rightarrow t = t_i = D' \alpha_i , i = 1, \ldots, N \]

The information for Pigou tax and command and control is the same.

Pro et con for tax:

Pro: dynamic effects of technical change.

Con: Administering a tax system based on individual rates, question of whether this will be seen as fair. Dynamic problems with too many firm closures, too few entries of new firms.