ECON4910 Environmental Economics
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More detailed solution to ex. 2 - Seminar 1

Ex. 2 Efficient provision of a public good

- Two agents, A and B,
- Private good: $x_i = x_A + x_B$
- Public good: $G = g_A + g_B$
- Consumer budget: $w_i = g_i + x_i \; \forall i \in \{A, B\}$

1. Define a Pareto optimal resource allocation.

   A Pareto optimal resource allocation is one in which there is no possible reallocation which makes one agent better off without making at least one other agent worse off. The Pareto optimal level of the public good is where the sum of all marginal rate of substitutions (MRS) equals the marginal rate of transformation (which is indirectly linked to the MC) $\sum MRS = MRT$

   This exercise compare two different utility functions in a public good contribution game, can you spot what differences it makes?

2. log-separable Utility: $u_i(G, x_i) = \alpha_i \ln G + \ln x_i$

   Social planner solution:

$$\max_{G, x_A, x_B} \quad \alpha_A \ln(G) + \ln(x_A)$$
$$s.t. \qquad \begin{cases} x_A + x_B + G = w_A + w_B \\ \alpha_B \ln(G) + \ln(x_B) = \overline{U}_B \end{cases}$$

The Lagrangian[1] and three first order conditions are

$$\mathcal{L} = \alpha_A \ln(G) + \ln(x_A) - \lambda_A(x_A + x_B + G - w_A + w_B) + \mu_A(\alpha_B \ln(G) + \ln(x_B) - \overline{U})$$
$$\frac{\partial \mathcal{L}}{\partial G} = \frac{\alpha_A}{G} - \lambda_A + \mu_A \frac{\alpha_B}{G} = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_A} = \frac{1}{x_A} - \lambda_A = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_B} = -\lambda_A + \mu_A \frac{1}{x_B} = 0$$

Combining we find:

$$\frac{1}{x_A} = \mu_A \frac{1}{x_B}$$
$$\Updownarrow$$
$$\frac{x_B}{x_A} = \mu_A$$

we find

$$\frac{\alpha_A}{G} = \frac{1}{x_A} + \frac{x_B}{x_A} \frac{\alpha_B}{G}$$
$$\Updownarrow$$
$$G^* = \alpha_A x_A + \alpha_B x_B$$

Which is the Pareto optimality condition MRS_A+ MRS_B = MC directly.

Note that marginal cost is 1 because the "production function" of the public good is just $G = g_A + g_B$. MRS with utility function $\alpha_A \ln(G) + \ln(x_A)$ is given by:

$$\text{MRS}_i = \frac{\partial u_i / \partial G}{\partial u_i / \partial x_i} = \frac{a_i x_i}{G}$$

and using condition (4.15):

$$\overbrace{\frac{a_A x_A}{G}}^{\text{MRS}_A} + \overbrace{\frac{a_B x_B}{G}}^{\text{MRS}_B} = \overbrace{1}^{\text{MC}}$$
$$\Updownarrow$$
$$G^* = a_A x_A + a_B x_B$$

[1]Why do i put a negative sign in front of λ_A but a positive sign in front of μ_A? Think about how relaxing the constraints, i.e. increasing total income or increasing the fixed utility level $\overline{U}$ would affect agent A's maximum utility.

Only feasible solutions are considered, i.e. the ones for which

$$G^* + x_A + x_B = w_A + w_B \tag{$*$}$$

Note that the solution for G^* is not unique; it depends on consumption levels x_A and x_B, which again depend on their initial endowments w_A and w_B.

3. Quasilinear utility function: $u_i(G, x_i) = \beta_i \ln G + x_i$

   Social planner solution:

$$\max_{G, x_A, x_B} \quad \beta_A \ln(G) + x_A$$

$$s.t. \qquad \begin{cases} x_A + x_B + G = w_A + w_B \\ \beta_B \ln(G) + x_B = \overline{U}_B \end{cases}$$

   The Lagrangian

$$\mathcal{L} = \beta_A \ln(G) + x_A - \lambda_A(x_A + x_B + G - w_A + w_B) + \mu_A(\alpha_B \ln(G) + x_B - \overline{U})$$

$$\frac{\partial \mathcal{L}}{\partial G} = \frac{\beta_A}{G} - \lambda_A + \mu_A \frac{\beta_B}{G} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_A} = 1 - \lambda_A = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_B} = -\lambda_A + \mu_A = 0$$

   Then $\lambda_A = 1$ and $\mu_A = 1$ gives:

$$\frac{\beta_A}{G} + \frac{\beta_B}{G} = 1$$

   which gives the social optimal level in a scenario with Quasi-linear utility:

$$MRS_A + MRS_B = MC \quad \rightarrow \quad \beta_A + \beta_B = \tilde{G}$$

4. Find the best response functions for g_A and g_B when applying the utility function: $u_i(G, x_i) = \alpha_i \ln G + \ln x_i$:

$$\max_{g_i} \ \alpha_i \ln(g_A + g_B) + \ln x_i$$

$$s.t. \ \begin{cases} x_i = w_i - g_i \\ g_i \geq 0 \end{cases}$$

The Lagrangian:

$$\mathcal{L} = \alpha_i \ln(g_A + g_B) + \ln(w_i - g_i) - \lambda_i(-g_i)$$
$$\frac{\partial \mathcal{L}}{\partial g_i} = \frac{\alpha_i}{g_A + g_B} + \frac{1}{x_i}(-1) + \lambda_i = 0$$

and the complementary slackness condition

$$\lambda_i g_i = 0$$

> *Note that the complementary slackness condition implies that either $g_i = 0$ or $\lambda_i = 0$ or both are zero at the same time. We make extensive use of this condition when determining the Nash equilibrium below, so keep it in mind!*

Case I) For an interior solution, $[\lambda_i = 0$ and $g_i > 0]$ then

$$g_i = \alpha_i x_i - g_j \quad \forall i \neq j$$

insert budget constraint:

$$g_i = \alpha_i(w_i - g_i) - g_j$$
$$g_i^{BR} = \frac{1}{1+\alpha_i}(\alpha_i w_i - g_j) \quad \forall i \neq j$$

Case II) Corner solution, either $[\lambda_i > 0$ and $g_i = 0]$, or $[\lambda_i = 0$ and $g_i = 0]$, then:

$$g_i^{BR} = 0$$

Solution:

$$g_i^{BR} = \max\left\{\frac{1}{1+\alpha_i}(\alpha_i w_i - g_j), 0\right\}$$

The best response of individual i depends on his initial wealth, w_i, his preferences α_i and it is decreasing in the others contribution g_j.

We see that individual i will **not** contribute if the other contributes enough $\alpha_i w_i < g_j$, then individual i prefers to be a free rider ($g_i^{BR} = 0$).

5. Find the best response functions for g_A and g_B when applying the quasilinear utility function: $u_i(G, x_i) = \beta_i \ln G + x_i$:

$$\max_{g_i} \; \beta_i \ln(g_A + g_B) + x_i$$
$$s.t. \begin{cases} x_i = w_i - g_i \\ g_i \geq 0 \end{cases}$$

The Lagrangian:

$$\mathcal{L} = b_i \ln(g_A + g_B) + (w_i - g_i) + \lambda_i g_i$$
$$\frac{\partial \mathcal{L}}{\partial g_i} = \frac{\beta_i}{g_A + g_B} - 1 + \lambda_i = 0$$

and the complementary slackness condition

$$\lambda_i g_i = 0$$

Case I) For an interior solution, $[\lambda_i = 0 \text{ and } g_i > 0]$ then

$$g_i^{BR} = \beta_i - g_j \quad \forall i \neq j$$

Case II) Corner solution, either $[\lambda_i > 0 \text{ and } g_i = 0]$, or $[\lambda_i = 0 \text{ and } g_i = 0]$, then:

$$g_i^{BR} = 0$$

Solution:

$$g_i^{BR} = \max\left\{\beta_i - g_j, 0\right\}$$

Notice: With the quasilinear utility function the contribution do **not** longer depend on wealth.

6. Explain when the agents decide to cooperate, and when it is optimal to free ride.

   - When $\beta_A = \beta_B$, there are an infinite number of Nash equilibria, but all of them involve $G^{NE} = \frac{\beta_A + \beta_B}{2}$. In particular, one Nash equilibrium is that both agents contribute $g_A = g_B = \frac{1}{2}\beta_A = \frac{1}{2}\beta_B$, but $g_A = 0$ and $g_B = \beta_B$ is also an equilibrium (check that the conditions are satisfied).
   - When $\beta_B > \beta_A > 0$, the Nash equilibrium is $g_B = \beta_B$ and $g_A = 0$. Thus the Nash equilibrium is unique as long as $\beta_A \neq \beta_B$. The agent with the highest preference for the public good will pay the whole cost of providing it, and the other agent will free-ride.
     We can write $G^{NE} = \max\{\beta_A, \beta_B\}$
   - When $0 < \beta_B < \beta_A$, the Nash equilibrium is reversed with $g_A = \beta_A$ and $g_B = 0$.
     We can write $G^{NE} = \max\{\beta_A, \beta_B\}$
   - (Draw these different equilibriums in a figure with the two BR-function. If the two BR-functions are crossing it is possible)

7. Compare the total level of the public good in the Nash equilibrium with the one in the Pareto allocation (compare G^{NE} in (6) and $\tilde{G}$ in (3))

   - if $\beta_A > \beta_B$ or $\beta_A > \beta_B$ then:

     $$G^{NE} = \max\{\beta_A, \beta_B\}$$

     which is always smaller than $\tilde{G} = \beta_A + \beta_B$
   - if $\beta_A = \beta_B$ then:

     $$G^{\mathrm{NE}} = \frac{1}{2}\beta$$

     which is always smaller than $\tilde{G} = \beta_A + \beta_B$