Problem Set 6 with Solution Key

4910 - Environmental Economics

1. Discounting.

Assume a CRRA utility function with $u_t = 2\sqrt{c_t}$ and suppose consumers maximize $\sum_{t=1}^{\infty} \delta^t u_t$ with $\delta = 0.9$.

(i) What is the discount rate on utility?

Solution key: It is given by ρ where $e^{-\rho t} = (0.9)^t$, so $\rho = -\ln(9/10) = 0.10536$, or 10.536 percent.

(ii) How much should one discount future consumption, if the growth rate of consumption is 2% a year?

Solution key: Apply Ramsey's formulae from the slides, with $\eta = 1/2$ and $\mu = 0.02$ to get:

$$r = \rho + \eta\mu = 0.10536 + \frac{1}{2}0.02 = 0.11536,$$

or 11.536 $\approx$ 12 percent.

(iii) Suppose there are two groups in the society. Half of the population are patient and have $\delta = 0.99$, while the other half is applying discount factor $\delta = 0.90$. Suppose you want to maximize the sum of todays' welfare (present-discounted value). What is the max amount you, as the planner, would be willing to invest/pay today if the value is worth 100 consumption units in 50 years? Which annual discount rate does this correspond to?

Solution key: When $\delta = 0.99$, then $\rho = -\ln(99/100) = 0.01005$, so

$$r = \rho + \eta\mu = 0.01005 + \frac{1}{2}0.02 \approx 0.02.$$

Thus, the aggregate value of 100 consumption units is:

$$100\left(\frac{1}{2}e^{-0.02*50} + \frac{1}{2}e^{-0.12*50}\right) = 100\left(\frac{1}{2}\left(\exp(-1) + \exp(-6)\right)\right) = 18.518.$$

This corresponds to an annual discount rate on consumption equal to R, where

$$\begin{aligned} 100e^{-R*50} &= 18.518, \text{ so} \\ R &= -\frac{\ln\left(\frac{18.518}{100}\right)}{50} = 0.03729, \end{aligned}$$

or 3.4 percent.

(iv) What is the answers to (iii) if instead the 100 consumption units are materialized in 100 years, not 50 years?

Solution key: In this case, the aggregate value of 100 consumption units is:

$$100\left(\frac{1}{2}e^{-0.02*100}+\frac{1}{2}e^{-0.12*100}\right)=100\left(\frac{1}{2}\left(\exp\left(-2\right)+\exp\left(-12\right)\right)\right)=6.7671.$$

This corresponds to an annual discount rate on consumption equal to R, where

$$\begin{aligned} 100e^{-R*100} &= 6.7671\text{, so} \\ R &= -\frac{\ln\left(\frac{6.7671}{100}\right)}{100}=0.026931, \end{aligned}$$

or 2.7 percent. Importantly, both discount rates are closer to the smallest one of 2 percent rather than the highest one of 12 percent, especially when the time horizon is 100 years: In this case, the smallest discount rate almost dictates the outcome. Thus, the optimal discount rate used by the planner decreases in the time horizon.

Note that the math from Weitzman '98 is consistent with two interpretations: Experts may have different views (as emphasized in the class), or individuals may have different preferences (as emphasized here). Using the notation from the class, p_i could be the fraction (rather than the probability) of the population who prefers to use discount rate r_i.

(v) Discuss informally how you think your answers would change if there is uncertainty about the consumption growth rate.

Solution key: We know that uncertain growth rate increases the risk and reduces the optimal discount rate one should use.

If the growth rate is not iid over time, but correlated, then the uncertainty is particularly large for long-term decisions /large time horizon.

Thus, this gives another motivation for applying a smaller discount rate for long-term decisions.

(vi) Also discuss how the answers would change if the consumption growth rate is certain, but it is quite uncertain whether the investment will give 100 consumption units.

Solution key: With risk aversion, a risky investment is less valuable so one is willing to invest less today to achieve that future consumption benefit. This suggest that one should use a higher discount rate when the rate of return to investment is uncertain, even though one should use a smaller discount rate when the growth rate of consumption is uncertain. You may want to emphasize these differences!

2. Time inconsistency.

Suppose a generation can choose its consumption level by maximizing $u_0 + \beta \sum_{t=1}^{\infty} \delta^t u_t$, where $\beta = \delta = 0.9$. Suppose there is only three periods, period 0, 1, and 2, and $u_t = c_t - \frac{1}{2} G_t^2$, where $G_t = G_{t-1} + c_t$ and $G_{-1} = 0$.

(i) If one can commit in period 0 to all future consumption levels, what is the optimal c_t for each period?

Solution key:

$$\max_{c_0, c_1, c_2} \left[c_0 - \frac{1}{2} c_0^2 + \beta\delta \left(c_1 - \frac{1}{2} (c_0 + c_1)^2 \right) + \beta\delta^2 \left(c_2 - \frac{1}{2} (c_0 + c_1 + c_2)^2 \right) \right],$$

which gives the three FOCs:

$$\begin{aligned} 1 - c_0 - \beta\delta (c_0 + c_1) - \beta\delta^2 (c_0 + c_1 + c_2) &= 0, \\ \beta\delta (1 - (c_0 + c_1)) - \beta\delta^2 (c_0 + c_1 + c_2) &= 0, \\ \beta\delta^2 (1 - (c_0 + c_1 + c_2)) &= 0. \end{aligned}$$

Solving these backwards:

$$\begin{aligned} c_0 + c_1 + c_2 &= 1, \\ c_0 + c_1 &= 1 - \delta (c_0 + c_1 + c_2) = 1/10, \\ c_0 &= 1 - \beta\delta (c_0 + c_1) - \beta\delta^2 (c_0 + c_1 + c_2) \\ &= 1 - \frac{81}{100} \left(\frac{1}{10} \right) - \frac{81}{100} \frac{1}{10} = \frac{838}{1000}, \text{ so} \\ c_1 &= \frac{1}{10} - c_0 = \frac{100 - 838}{1000} = -\frac{738}{1000}, \text{ and} \\ c_2 &= 1 - (c_0 + c_1) = \frac{900}{1000}. \end{aligned}$$

Interpretation: In the first period, one pollutes a lot because of time inconsistency. But for the next period, one plans to clean up for the benefit of the last period. In the last period, it is optimal to pollute a lot because it is the last period.

(ii) If one cannot commit in period 0, what is the equilibrium values of c_t for each period?

Solution key:

In the last period, we get, as before: $c_0 + c_1 + c_2 = 1$. In period $t = 1$, that decision maker's problem is:

$$\begin{aligned} & \max_{c_1} \left[\left(c_1 - \frac{1}{2} (c_0 + c_1)^2 \right) + \beta\delta \left(c_2 - \frac{1}{2} (c_0 + c_1 + c_2)^2 \right) \right] \\ = \; & \max_{c_1} \left[\left(c_1 - \frac{1}{2} (c_0 + c_1)^2 \right) + \beta\delta \left(1 - (c_0 + c_1) - \frac{1}{2} \right) \right], \end{aligned}$$

which gives the FOC:

$$\begin{aligned} 1-(c_0+c_1)-\beta\delta &= 0, \text{ or} \\ c_0+c_1 &= 1-\frac{81}{100}=\frac{19}{100}, \text{ so} \\ c_1 &= \frac{19}{100}-c_0. \end{aligned}$$

Thus, in period 0:

$$\begin{aligned} &\max_{c_0}\left[c_0-\frac{1}{2}c_0^2+\beta\delta\left(c_1-\frac{1}{2}(c_0+c_1)^2\right)+\beta\delta^2\left(c_2-\frac{1}{2}(c_0+c_1+c_2)^2\right)\right] \\ =\;&\max_{c_0}\left[c_0-\frac{1}{2}c_0^2+\left(\frac{9}{10}\right)^2\left(\frac{19}{100}-c_0-\frac{1}{2}\left(\frac{19}{100}\right)^2\right)+\left(\frac{9}{10}\right)^3\left(1-\frac{19}{100}-\frac{1}{2}\right)\right], \end{aligned}$$

so the FOC becomes:

$$\begin{aligned} 1-c_0-\frac{81}{100} &= 0, \text{ so} \\ c_0 &= \frac{19}{100}, \\ c_1 &= 0, \\ c_2 &= \frac{81}{100}. \end{aligned}$$

So, one pollutes more (or does not clean) in period 1 when one cannot commit, motivating less pollution at time 2 as well as at time 0.

(iii) If one cannot comit in period 0, will generation 0 choose c_0 strategically? In which sense, and to influence which choice?

Solution key: Yes: The larger is c_0, the smaller is c_1. At time 0, the decision maker would like to commit to more cleaning at time 1, and at least one can induce less consumption at time 1 by consuming strategically more at time 0.