Self insurance with complete markets

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We consider an infinitely lived agent with discount factor $\beta < 1$ and utility function $u(\cdot)$, who has an income stream y_t and can borrow and lend at a rate of interest r. Income today is known, but future income is stochastic.

The agent maximizes

$$E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

subject to the set of budget constraints

$$c_t + s_t = y_t + (1+r)\, s_{t-1}. \tag{BC}$$

Here s_t is the amount of savings put aside in period t for use in period $t+1$. There is one budget constraint for each period t.

The expected total lifetime utility from period 0 onwards given savings s_{-1} is

$$\begin{aligned} V(s_{-1}) &= \max_{c_0, c_1, c_2, \ldots} E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right] \text{ s.t. (BC)} \\ &= \max_{c_0, c_1, c_2, \ldots} \left[u(c_0) + \beta E\left(\sum_{t=1}^{\infty} \beta^{t-1} u(c_t)\right)\right] \text{ s.t. (BC)} \end{aligned}$$

As the last term of the second line is similar to the first line, we get

$$V(s_{-1}) = \max_{c_0, s_0} u(c_0) + \beta E[V(s_0)] \text{ s.t. (BC)},$$

and as it follows from (BC) that

$$c_0 = y_t + (1+r)\, s_{t-1} - s_0,$$

we get

$$V(s_{-1}) = \max_{c_0, s_0} u[y_t + (1+r)\, s_{t-1} - s_0] + \beta E[V(s_0)].$$

The first order condition yields

$$u'(c_0) = \beta E\left[V'(s_0)\right],$$

and

$$V'(s_{-1}) = (1+r)\,u'(c_0) + \left[-u(c_0) + \beta E\left[V'(s_0)\right]\right]\frac{\partial s_0}{\partial s_{-1}}.$$

As the term in square brackets is zero from the first order condition (or from the envelope theorem), we get

$$V'(s_{-1}) = (1+r)\,u'(c_0).$$

The same condition holds for every period, so

$$V'(s_0) = (1+r)\,u'(c_1).$$

Combining this with the FOC yields

$$u(c_0) = \beta(1+r)\,E\left[u'(c_1)\right],$$

or more generally

$$u(c_t) = \beta(1+r)\,E\left[u'(c_{t+1})\right].$$

This expression is often called the *Euler equation.* To find the level of consumption, notice that if we add (BC) muliplied by $(1+r)^t$ for every period and require savings to go to zero (the no Ponzi games condition), we get

$$\sum_{t=0}^{\infty}(1+r)^t c_t = \sum_{t=0}^{\infty}(1+r)^t y_t.$$

The right hand side of this expression is the discounted value of future income. Hence a change in y_t today only has an impact on c_t through the change in teh discounted future income, and it will have a similar impact on all future levels of consumption.