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On Intertemporal Optimization and Dynamic Efficiency: From discrete to continuous time (the maximum principle)[1]

This note might, hopefully, be some help in understanding some of the ("mysterious") conditions you see when solving a dynamic optimization problem, using Pontryagin's Maximum Principle ("control theory"). Before launching a discrete time version of the well-known Ramsey model which we turn into a continuous time framework, by taking limits, let us consider a simple three-period model, with capital equipment and some non-renewable resource, both used as inputs to generate a macro output to be used for consumption and gross investment. This set-up will introduce some important concepts and relevant rates of interest or rates of return we will use.

1. On dynamic efficiency and optimality

We have a three-period economy, with an initial stock of capital equipment, K_0, and an initial stock of a non-renewable natural resource, S_0. An output per period is produced by using the services of capital equipment along with resources (as a flow). In the first period, output is $F(K_0, R_1)$, where F is a standard neoclassical production function, obeying the Inada-conditions. This output is allocated to consumption in the first period, c_1, and gross investment $J_1 = K_1 - (1-\delta)K_0$, where a fraction $\delta \in (0,1)$ of the existing capital stock is depreciated per period. In addition, the remaining stock of the non-renewable resource at the end of the first period is given by $S_1 = S_0 - R_1$. Consumption per period is evaluated according to a one-period utility function, $U(c)$, which is strictly increasing, bounded and strictly concave; with $U'(0) = \infty$. The utility function is time-independent; the same in any period. (We assume, conventionally, that consumption is to take place at the beginning of any period.) Another assumption is that capital can without cost be transformed back to consumption (of course a rather special assumption, but it helps making the analysis simpler).

To sum up: In the first period we have: $F(K_0, R_1) = c_1 + K_1 - (1-\delta)K_0$ and $S_1 = S_0 - R_1$; where consumption is evaluated according to $U(c_1)$.

[1] This note might be read along with "Optimal control theory with applications to resource and environmental economics", by Michael Hoel.

In the second period we have: $F(K_1, R_2) = c_2 + K_2 - (1-\delta)K_1$ and $S_2 = S_1 - R_2$, with present discounted utility (i.e. the utility from consumption in the second period, evaluated from the beginning of the first period, is given by $\beta U(c_2)$, where β is a constant one-period utility discount factor; with $\beta \in (0,1]$, capturing pure impatience. Finally, in the last period 3, we have: $F(K_2, R_3) = c_3 + K_3 - (1-\delta)K_2$ and $S_3 = S_2 - R_3$, with a present discounted utility as given by $\beta^2 U(c_3)$, as seen from the beginning of the first period. Because period 3 is the last one, we impose some terminal constraints on the stock variables K_3 and S_3. Because there is no future after this period, we impose the following end-point or terminal constraints $K_3 \geq 0, S_3 \geq 0$. (Both constraints will in optimum be binding; welfare will be reduced if we don't exploit all available resources for consumption before the end of the planning period.)

If we insert for consumption in each period, along with the dynamics of the stock variables, we can express the present discounted utility, as a function of the state variables only, as:

$$V(K_1, K_2, K_3, S_1, S_2, S_3) := U(F(K_0, S_0 - S_1) - K_1 + (1-\delta)K_0) + \beta U(F(K_1, S_1 - S_2) - K_2 + (1-\delta)K_1)$$
$$+\beta^2 U(F(K_2, S_2 - S_3) - K_3 + (1-\delta)K_2)$$

Because there is no uncertainty or risk, a social planner can put up a strict plan from the beginning of the first period, in the sense that all variables can be determined optimally at this stage. (There is no incentive to revise the plan at a later stage; hence we do not face any problem with dynamic inconsistency in the sense that at the beginning of any period, the planner will take the same decision as she did initially, if she were allowed to re-optimize.)

The planner will now maximize her objective $V(K_1, K_2, K_3, S_1, S_2, S_3)$, subject to the terminal constraints $S_3 \geq 0$ and $K_3 \geq 0$. Given our assumptions the problem has a solution. Wecan then define the Lagrangean function as $L = V + \lambda S_3 + \mu K_3$, where $\lambda \geq 0$ and $\mu \geq 0$ are multipliers (shadow prices) put on the terminal stock variables.

An intertemporal optimal allocation must then obey the following conditions, which, given concavity, are sufficient as well, for characterizing an optimum:

$$(1-1) \qquad \frac{\partial L}{\partial K_1} = 0 \Leftrightarrow \frac{\partial V}{\partial K_1} = U'(c_1)(-1) + \beta U'(c_2)\left[\frac{\partial F}{\partial K_1} + 1 - \delta\right] = 0$$

for optimal values of the remaining variables. What is the interpretation of this first-order condition? In this period we have to balance or trade off current consumption against investment; higher consumption will crowd out investment in the sense that less capital will be transferred to the next period. Hence we have to trade off current benefit from higher consumption against the loss of future benefit due to lower capital and thus lower capacity for producing consumption goods in the next period. For a given (optimal) value of K_2 and S_1, the benefit of higher capital equipment to be used in the next period (period 2), which is the present value of marginal utility of consumption in period 2, caused by the corresponding increase in output in period 2, should balance the marginal loss in current utility caused by lower consumption in period 1. On lowering consumption by one unit in period 1, at a utility cost $U'(c_1)$, while leaving one more unit of capital to be used in the next period, total increase in output being available for consumption in period 2 per unit consumption in period 1 is then $1 + \frac{\partial F}{\partial K_1} - \delta$; i.e., the unit itself (that explains the number "1") plus the net marginal productivity of capital. The present value of the utility gain of this higher consumption in the next period is then $\beta U'(c_2)\left[\frac{\partial F}{\partial K_1} + 1 - \delta\right]$. Then we get (1-1).

In the next period, period 2, we have a similar trade-off, leading to the optimality condition:

$$(2-1) \qquad \frac{\partial V}{\partial K_2} = \beta U'(c_2)(-1) + \beta^2 U'(c_3)\left[1 + \frac{\partial F}{\partial K_2} - \delta\right] = 0$$

This condition has the same interpretation as (1-1) above, for constant (and optimally adjusted) K_1 and S_2.

What about the use of natural resource in the first period? The answer to this question is related to how much of the resource stock we want to hand over as input in the subsequent period. That is; what is the optimal value of S_1? This follows from:

$$(1-2) \qquad \frac{\partial L}{\partial S_1} = 0 \Leftrightarrow \frac{\partial V}{\partial S_1} = U'(c_1)(-\frac{\partial F}{\partial R_1}) + \beta U'(c_2)\frac{\partial F}{\partial R_2} = 0$$

The interpretation of this condition is: Higher stock of resources at the beginning of the second period; $S_1 = S_0 - R_1$, is provided by reducing the input flow in the first period through a lower R_1, which, along with a constant stock of capital equipment, will give lower consumption in the first period. If we reduce R_1 by one unit (and hence increase the stock available at the beginning of the next period, S_1, by one unit), output will be reduced by $\frac{\partial F}{\partial R_1}$ units of the consumption good, when capital equipment is kept fixed. The total marginal loss in the first period is then $U'(c_1)\frac{\partial F}{\partial R_1}$, which has to be balanced against the present value of discounted utility gain of higher consumption in the second period, realized by having a higher input flow of natural resources, for an optimally fixed value of S_2. The present value of this utility gain is $\beta U'(c_2)\frac{\partial F}{\partial R_2}$.

Along the same line, we can determine the optimal stock of the resources as being handed over to the last period. This has the same feature as (1-2) above, with a similar interpretation:

$$(2-2) \qquad \frac{\partial V}{\partial S_2} = -\beta U'(c_2)\frac{\partial F}{\partial R_2} + \beta^2 U'(c_3)\frac{\partial F}{\partial R_3} = 0$$

At last we have to determine the terminal values of the stocks to be left at the end of the last period. These are found respectively as

$$(3-2) \qquad \frac{\partial L}{\partial S_3} = -\beta^2 U'(c_3)\frac{\partial F}{\partial R_3} + \lambda = 0 \text{ and } \lambda S_3 = 0, \text{ med } \lambda \geq 0 \ (= 0 \text{ if } S_3 > 0)$$

$$(3-1) \qquad \frac{\partial L}{\partial K_3} = -\beta^2 U'(c_3) + \mu = 0 \text{ and } \mu K_3 = 0, \text{ med } \mu \geq 0 \ (= 0 \text{ if } K_3 > 0)$$

Given our assumptions, we have λ and μ both positive; hence the constraints are effectively binding with no real capital and nothing of the natural resource handed over to the future. (The conditions (3-1) and (3-2) are later associated with so-called *transversalilty conditions*.)

Therefore, for any period before the last one we must have:

$$(I) \qquad -U'(c_t) + \beta U'(c_{t+1})\left[\frac{\partial F}{\partial K_t} + 1 - \delta\right] = 0 \text{ for } t = 1,2$$

This condition says: The loss in utility caused by a unit reduction in consumption at the beginning of period t – or the current utility cost due to increased stock of capital available at the beginning of the subsequent period – is balanced against the present discounted increase in utility caused by having a higher supply of goods for consumption in period $t+1$. (The higher supply is made up of the net marginal productivity $\frac{\partial F}{\partial K_t} - \delta$ in addition to the capital unit itself, which can be consumed; hence the increased amount of output per unit capital equipment is $1 + \frac{\partial F}{\partial K_t} - \delta$.)

We can rewrite this expression as being read as equalizing the required rate of return from non-consumption ("saving") and the real rate of return on capital:

We know that the marginal rate of substitution (MRS) shows the maximal amount of consumption in period $t+1$ that is required to compensate for one unit consumption in period t, without any loss in utility. This MRS is defined as $-\frac{dc_{t+1}}{dc_t} = \frac{U'(c_t)}{\beta U'(c_{t+1})}$, which by assumption is strictly declining: In order to be willing to give up consumption today for higher consumption tomorrow, one needs a higher future compensation the less we are consuming today. The MRS shows the number of units of consumption in period t+1 per unit consumption in period t. To get a relative rate or a rate of interest, we consider $MRS - 1 = \frac{U'(c_t)}{\beta U'(c_{t+1})} - 1 = \frac{U'(c_t) - \beta U'(c_{t+1})}{\beta U'(c_{t+1})}$, which is a rate of interest, as defined as ρ_t, showing what rate of return you will require so as to be willing to give up consumption today without lowering utility. On using this definition in (I), we get:

$$(I)' \qquad -\frac{dc_{t+1}}{dc_t} - 1 = \frac{U'(c_t)}{\beta U'(c_{t+1})} - 1 := \rho_t = \frac{\partial F}{\partial K_t} - \delta := \theta_t$$

where we have defined the real rate of return on capital or «the own rate of return", denoted θ_t or the rate of interest per period in terms of the macro output.

We then have our first important conclusion for how to allocate assets over time: Within the present context, optimal capital accumulation (and hence the associated consumption in each period) is determined so that in any period, the required rate of return on non-consumption ("saving") should be equal to the real rate of return on capital ("investment").

Remark 1

We can define an interest rate for any commodity being traded in different periods of time. Suppose we have a market economy with a full set of forward markets, by having prices, quoted at the outset of the first period, for a commodity no. i, as a sequence $\left\{p_{i1}, p_{i2}, \ldots, p_{it}, ..\right\}$, showing what you have to pay at the outset of the first period for a unit of the commodity delivered in period t. These prices are called discounted or present value prices, and supposed to be quoted in some monetary unit. Given such a price structure, you will want to act according to the standard rule for utility maximization as a price taker, according to $\frac{U'(c_t)}{\beta U'(c_{t+1})} = \frac{p_{ct}}{p_{c,t+1}}$. Then we have an own rate of interest for this commodity over the period $\left[t, t+1\right]$, defined as: $\xi^c_{t,t+1} := \frac{p_{ct}}{p_{c,t+1}} - 1$.

On giving up one unit of consumption at t, you save an amount of money p_{ct}, at the beginning of the first period. For delivery in the subsequent period, each monetary unit saved in period t, can purchase $\frac{1}{p_{c,t+1}}$ units of additional consumption to be delivered at the beginning of $t+1$; hence, you can totally acquire $\frac{p_{c,t}}{p_{c,t+1}}$ additional units of consumption to be delivered at the beginning of period t+1, which is equivalent to the MRS as shown above. The relative rate of return, as a percentage, from this kind of intertemporal substitution, is $\xi^c_{t,t+1} := \frac{p_{ct}}{p_{c,t+1}} - 1$. This is the own rate of interest on consumption between t and t+1.

To take a step towards continuous time, let us consider the period between t and $t+h$, and define the own rate of interest on consumption in a period of time of length h, as the average rate of change in the price of consumption over this period, as given by $\frac{p_{c,t} - p_{c,t+h}}{hp_{c,t+h}} = \xi^c_{t,t+h}$. Let $h \downarrow 0$, and assume the limit exists. We then get $\xi^c(t) = -\frac{\dot{p}(t)}{p(t)}$, where $\dot{p}(t) := \frac{dp(t)}{dt}$ is the derivative with respect to time. The continuous own rate of interest per unit of time is equal the negative of the relative rate of change in the price of consumption. If the price of consumption is expected to decline, then the own rate of interest is positive as the instantaneous rate of return from giving up consumption at t is positive. On the other hand, if the price increases, then the own rate of interest is negative, as you will experience a loss from delaying consumption "from *t* to $t+h$" as seen at point in time t.

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Because there are two assets in this economy, producible real capital equipment and a non-renewable natural resource, optimal wealth management must also include optimal holding of the resource as a stock, and the corresponding use of the resource per period (as a flow). Resource saving now means that we are not extracting all resources today, but instead leave some of it in the ground available for subsequent periods. What is being used in one period, is used up physically, and cannot be used again. That is the true character of a non-renewable resource. Current use will therefore have an opportunity cost reflecting the value of the resource used for future production. The value of current use or consumption (as given by the "rental value" or the price per unit extracted of the resource, in units of utility) has to balance the future return on the remaining reserve as an asset (saving), given by a capital gain. By delaying extraction or choosing not to extract one unit of the resource today, one might use this unit tomorrow to produce goods for consumption or for capital accumulation. We have seen that the trade-off with respect to the amount of the resource to be left over to the future, from which we can derive how much of the resource to be extracted per period (except the last one), is given by the condition

$$(II) \qquad U'(c_t)(-\frac{\partial F}{\partial R_t}) + \beta U'(c_{t+1})\frac{\partial F}{\partial R_{t+1}} = 0 \ \text{ for } t = 1,2$$

which has been interpreted earlier. However, this condition can be rewritten in the following two alternative ways: One is to express (II) as

$$(II)' \quad MRS(c_t, c_{t+1}) = \frac{U'(c_t)}{\beta U'(c_{t+1})} = \frac{\dfrac{\partial F}{\partial R_{t+1}}}{\dfrac{\partial F}{\partial R_t}} \ \text{ for } t = 1,2$$

where the RHS of (II)′ is a marginal rate of transformation between output in period $t+1$ and in period t, for the resource input. For optimal capital equipment, the number of consumption units required for period $t+1$ for giving up one unit of consumption in period t coincides with the same rate at the production side, showing the gain from saving an amount of resource required for producing one unit of output in period *t* to be used in the production process in period *t+1*. Or, we can transform (II) into an equality between the required rate of return from non-consumption and the real rate of return from resource saving or delaying extraction of the non-renewable natural resource.

From (II)′ we get, when using the definition of the required rate of return from non-consumption in period t, as defined in (I)′, we get:

$$(II)'' \quad \frac{U'(c_t)}{\beta U'(c_{t+1})} - 1 = \rho_t = \frac{\dfrac{\partial F}{\partial R_{t+1}} - \dfrac{\partial F}{\partial R_t}}{\dfrac{\partial F}{\partial R_t}} := \sigma_t$$

The rate of change in marginal productivity of the resource use between two subsequent periods, is the rate of return on the resource stock (in ground) considered as an asset; defined here as σ_t. This is a rate of interest per period in units of consumption. (The marginal productivity of the resource can be seen as the price for the resource, in terms of the consumption good. Hence, the RHS shows the rate of change in the resource price, or a capital gain on the unextracted resource between the two periods.) We therefore have:

$$(III) \quad \theta_t = \rho_t = \sigma_t \Leftrightarrow \frac{\partial F}{\partial K_t} - \delta = \frac{U'(c_t)}{\beta U'(c_{t+1})} - 1 = \frac{\frac{\partial F}{\partial R_{t+1}} - \frac{\partial F}{\partial R_t}}{\frac{\partial F}{\partial R_t}} \text{ for t=1,2}$$

An intertemporal optimum or optimal wealth management is therefore characterized by equalizing the return on capital equipment (or own rate of return) and the rate of change in the marginal productivity of the natural resource (or capital gain), and both being equal to the required rate of return from non-consumption, all rates measured in units of consumption.

Remark 2:
The condition $\theta_t = \sigma_t$ for $t = 1,2,3$, will characterize an intertemporally efficient programme. Given the technologies, initial and (binding) terminal constraints on resources and capital, we then have an allocation such that it is not possible to increase consumption in some period without reducing it in some other period. (This is similar to static efficiency. The allocation is a point on the boundary of the consumption possibility set which is the collection of all efficient programmes.) On equalizing this efficiency condition with the required rate of return, we determine the optimal point on this boundary. Hence, on maximizing intertemporal utility among all efficient programmes, we get the dynamic optimum.

Remark 3:
When we go to continuous time, the "resource rent" or the rate of return from delaying extraction is given as $\frac{\frac{d(\frac{\partial F}{\partial R_t})}{dt}}{\frac{\partial F}{\partial R_t}} = \frac{d \ln F_R}{dt}$ which is the continuous counterpart to our discrete version above $\frac{\frac{\partial F}{\partial R_{t+1}} - \frac{\partial F}{\partial R_t}}{\frac{\partial F}{\partial R_t}}$.

2. Towards the Maximum Principle

We consider the well-known Ramsey model for optimal saving (with only producible real capital) and start with discrete time, and let, for a moment, each period has a length of time equal to one unit of time (say; one year). We assume that output per period (per year) – a flow – is generated from a standard neoclassical production function (strictly increasing and concave, and with $F'(0)$ sufficiently high), depending only on the stock of capital present at the beginning of the period. We now ignore the presence of any other input; also natural resources. (Capital is reversible in the sense that it can without cost be transformed to consumption, if necessary.) The output in any period is used for consumption and gross investment.

Consider period *t*, for which we have the following balancing condition: Output in period t is allocated to consumption and/or gross investment:

$$(1)\quad F(K_{t-1}) = C_t + J_t = C_t + K_t - K_{t-1} + \delta K_{t-1} \quad \text{for } t = 1, 2, \ldots.$$

Here output per year, $F(K_{t-1}) \cdot 1$, is used for consumption C_t and gross investment $J_t = K_t - K_{t-1} + \delta K_{t-1}$, given by the sum of net investment $(K_t - K_{t-1})$ in period *t*, and depreciation of capital equipment during the period (a constant fraction $\delta \in (0,1)$ of the stock of capital present at the outset of the period.)

Let us now divide each period in n sub-periods of equal length h units of time. (Think of the year as 12 months, each of length 30 days.) Then each sub-period in year *t* can be represented by a time interval, say two subsequent days in August in the year 2016, $\left[t + \frac{i}{n}, t + \frac{i+1}{n}\right]$, for $t = 0, 1, \ldots, T-1$ and for $i = 0, 1, \ldots, n-1$.

Let us then consider gross output per sub-period. With a capital equipment $K_{t+\frac{i}{n}}$ we could have produced, with this capital equipment, over the whole year a gross output $F(K_{t+\frac{i}{n}}) \cdot 1$, where the number «1» represents the length of a year.[2] However, we want

[2] The multiplication by the number «one» shows the very important distinction between flows and stocks. Capital is a stock variable, given by the number of machines present *at* some point in time. Output per unit of time at some point in time is a flow variable; really an intensity, must be multiplied by the length of the time period over which the flow operates to get an output. It is the same as saying that if you drive a car at a speed 80 km per hour, you don't get any distance without multiplying this speed with some

to have a measure of output per sub-period; say per day. This will then be given by the output per unit of time $F(K_{t+\frac{i}{n}})$, the flow, multiplied by the length of the time period under consideration, as given by $\frac{1}{n} := h$. Hence output per sub-period is then $F(K_{t+\frac{i}{n}}) \cdot h$, which is used for consumption in the period $t + \frac{i+1}{n}$, as given by $C_{t+\frac{i+1}{n}}$, and gross investment as defined as $J_{t+\frac{i+1}{n}} = K_{t+\frac{i+1}{n}} - K_{t+\frac{i}{n}} + \frac{\delta}{n} K_{t+\frac{i}{n}}$, when taking into account that the depreciation per sub-period is $\frac{1}{n}$ of the annual depreciation rate.

Consumption during any sub-period is defined as a constant consumption intensity or flow, multiplied by the length of the time period. Let the consumption intensity per unit of time be c_τ with consumption during a short interval $\Delta\tau$, given by $c_\tau \Delta\tau$, with c_τ being constant, by assumption. Then we have that consumption during the period $\left[t + \frac{i}{n}, t + \frac{i+1}{n}\right]$ can be defined as «the continuous sum of all flows, $c_{t+\frac{i}{n}}$ during this time period", given by the integral:

$$(2) \quad C_{t+\frac{i+1}{n}} = \int_{t+\frac{i}{n}}^{t+\frac{i+1}{n}} c_\tau d\tau = h c_{t+\frac{i}{n}}$$

Suppose that the instantaneous utility produced by some consumption intensity c_τ, is $U(c_\tau)$, which is increasing and strictly concave, with $U'(0) = \infty$. During a period of length h, the utility flow is then $U(c_\tau)h$.

If the annual utility discount rate or utility time preference rate («the felicity rate») is $100r\%$, the discount rate per sub-period is $\frac{r}{n} = rh$. Define then a *discount factor* for the new accounting period as $\beta := \frac{1}{1+rh}$.

measure of time itself; say one hour which gives you a distance of 80 km. I think this very important issue is under-communicated among current generations of students in economics.

We should then be able to formulate a meaningful optimization problem for a social planner whose objective is to choose a consumption path and a corresponding path of real capital equipment, so as to maximize the present discounted total utility from today on and to a fixed horizon (or a fixed number of future periods), where the objective function can be written as $\sum_{j=0}^{nT-1} \beta^{j+1} U(c_{j+1})h$, for a total of nT sub-periods, given that:

$$(1)' \quad F(K_j)h \geq c_{j+1}h + K_{j+1} - (1-\delta h)K_j \quad j = 0,1,\ldots,nT-1$$

(We drop $\geq$, and put equality sign instead, because we will never operate with a slack.)

(3) $K_0 \leq A$ (A being a given initial capital equipment), $K_j \geq 0, c_j \geq 0$; $j = 1,2,\ldots,nT$.

Given our standard assumptions, this problem has an interior solution, expressed as vectors, $(\hat{c}, \hat{K}, \hat{\lambda})$, with a set of non-negative numbers (present discounted values of the Lagrange multipliers), $\lambda := \left(\lambda_1, \lambda_2, \ldots, \lambda_{nT}\right)$, with consumption flows and capital equipment in each period obeying the first-order conditions:

$$(4) \quad \frac{\partial L}{\partial c_{j+1}} = \beta^{j+1} U'(\hat{c}_{j+1})h - \lambda_{j+1}h = 0 \quad \text{for } j = 0,1,\ldots,nT-1$$

$$(5) \quad \frac{\partial L}{\partial K_j} = -\lambda_j + \lambda_{j+1}\left[1 + hF'(\hat{K}_j) - \delta h\right] = 0 \text{ for } j = 0,1,\ldots,nT-1$$

$$(6) \quad \frac{\partial L}{\partial K_{nT}} = -\lambda_{nT} \leq 0 \quad (= 0 \text{ if } \hat{K}_{nT} > 0) \text{ and } \hat{K}_0 \leq A \ \ (= A \text{ if } \lambda_0 > 0)$$

where the Lagrangian function is

$$(7) \quad L(c,K,\lambda) = \sum_{j=0}^{nT-1} \beta^{j+1} U(c_{j+1})h - \lambda_0\left[K_0 - A\right] - \sum_{j=0}^{nT-1} \lambda_{j+1}\left[c_{j+1}h + K_{j+i} - (1-\delta h)K_j - hF(K_j)\right]$$

If we drop the «hats» for the optimal solution, we observe that we can write the optimality conditions (with $\lambda >> 0$, $c >> 0$, and $K > 0$ with $K_0 = A$ and $K_{nT} = 0$) as:

$$(4)' \quad \beta^j U'(c_j) = \lambda_j \quad j = 1,\ldots,nT$$

$$(5)' \quad \lambda_{j+1}\left[1 + hF'(K_j) - \delta h\right] = \lambda_j \qquad j = 0,1,\ldots,nT-1$$

$$(1)' \quad F(K_j)h = c_{j+1}h + K_{j+1} - (1-\delta h)K_j \qquad j = 0,1,\ldots,nT-1$$

Consider then the continuous counterpart to (4)′ and (1)′ first, and start with (1)′.[3]

On dividing through (1)′ by h, we get: $\dfrac{K_{t+(i+1)h} - K_{t+ih}}{h} = F(K_{t+ih}) - c_{t+(i+1)h} - \delta K_{t+ih}$.

Let $h \downarrow 0$, and we get the continuous counterpart to our balancing condition:

$$(1)'' \qquad \dot{K}(t) = F(K(t)) - c(t) - \delta K(t) \Leftrightarrow F(K(t)) = c(t) + \dot{K}(t) + \delta K(t)$$

To get to the continuous counterpart to (4)′ we use the following definition of the natural logarithm to get that interest now compounds exponentially:

$$e^{-rt} = \lim_{h\to 0} \beta^{t+ih} = \lim_{n\to\infty} \left(\frac{1}{1+\frac{r}{n}}\right)^{t+\frac{i}{n}}$$

so that:

$$(4)'' \quad e^{-rt}U'(c(t)) = \lambda(t)$$

The present discounted value of marginal utility of consumption, in units of utility (utils) – a short-term effect – is equal to the value of the shadow price or what it is sometimes called, the adjoint or costate variable $\lambda(t)$, which can be interpreted as the "future" or marginal long-run impact of the current decision, measured in utils. From (5)′, when using (4)′, we get the first interpretation: On giving up one unit of consumption in sub-period *j*, so as to increase the stock of capital with one unit at the beginning of the subsequent sub-period, the utility cost is λ_j. This cost must be balanced, all other things being optimally adjusted, against the utility gain of the additional output for consumption that will be provided by the higher capital equipment in the future made possible through lowering consumption now. The amount of gross output being available for consumption is then $1 + h(F'(K_j) - \delta)$. The

[3] We assume all limits exist.

utility evaluation of each marginal unit of subsequent consumption is λ_{j+1}; hence we get $\lambda_{j+1}\left[1+hF'(K_j)-\delta h\right]=\lambda_j$. This condition can be expressed as

$\lambda_{t+(i+1)h}[F'(K_{t+ih})-\delta]=\dfrac{\lambda_{t+ih}-\lambda_{t+(i+1)h}}{h}$, where $\lambda_{t+ih}-\lambda_{t+(i+1)h}$ is a measure of the loss in the value of capital, in units of utility, from period $t+ih$ to $t+(i+1)h$, or depreciation of capital. On letting $h\downarrow 0$, we get the continuous version:

$$(5)'' \quad -\dot{\lambda}(t)=\lambda(t)\left[F'(K(t))-\delta\right]\Leftrightarrow F'(K(t))-\delta=-\frac{\dot{\lambda}(t)}{\lambda(t)}$$

We have seen before that $F'(K(t))-\delta$ is a real rate of return (an interest rate) on capital, and from Remark 1, we can identify the relative rate of decline in the shadow price also as a rate of interest.

At last, by noting that the limit of the objective function can be written as an integral, because we have: $\lim_{h\to 0}\sum_{j=0}^{nT-1}\beta^{j+1}U(c_{j+1})h=\int_0^{T^*}e^{-rt}U(c(t))dt$, with T^* as the corresponding end-point of the planning period.

The continuous counterpart to our previous dynamic optimization problem is then:

Problem-continuous

$$\textit{Maximize} \quad \int_0^{T^*}e^{-rt}U(c(t))dt$$

$$\textit{such that} \quad \dot{K}(t)=F(K(t))-\delta K(t), K(0)=A,\ K(T^*)=0$$

$$\textit{with} \quad K(t)\geq 0, c(t)\geq 0\ \forall t\in\left[0,T^*\right],\ T^* \textit{ fixed}$$

First we suppose that the integral exists, so that our problem has a solution. Then the task is to choose a consumption path and an associated capital path, which has to evolve over time according to the differential equation above, so that the present discounted utility or the integral is maximized over the fixed planning period, given the dynamics of the capital equipment, with some initial capital equipment and a given requirement on terminal capital. This is a problem of optimal control, with c being the control variable, whereas K is called a state variable. The utility discount rate r in the objective function must here be regarded as a rate of pure time preference or a rate of impatience.

We can try to formulate this problem first as a Lagrangian problem (but now we must be cautious as there are an infinite number of side constraints as given by the differential equation for the state variable). Define a Lagrangian function L, with corresponding Lagrangian multipliers, as given by:

$$L = \int_0^{T^*} \left\{ e^{-rt}U(c(t)) + \lambda(t)\left[F(K(t)) - c(t) - \delta K(t) - \dot{K}(t)\right]\right\} dt - \mu\left[K(0) - A\right] + \vartheta K(T^*)$$

$$= \int_0^{T^*} \left\{ e^{-rt}U(c(t)) + \lambda(t)\left[F(K(t)) - c(t) - \delta K(t)\right]\right\} dt - \int_0^{T^*} \lambda(t)\dot{K}(t)dt - \mu\left[K(0) - A\right] + \vartheta K(T^*)$$

where μ, ϑ are non-negative multipliers imposed on the initial and terminal constraints on the capital stock, respectively. Before we look at the partial derivatives of the L-function with respect to c, K, as we do in standard static optimization problems with side-constraints, we want to «get rid» of the troublesome $\dot{K}(t)$. On integrating by parts we get:

$$\int_0^{T^*} \lambda(t)\dot{K}(t)dt = \left[\lambda(t)K(t)\right]_0^{T^*} - \int_0^{T^*} \dot{\lambda}(t)K(t)dt = \lambda(T^*)K(T^*) - \lambda(0)K(0) - \int_0^{T^*} \dot{\lambda}(t)K(t)dt$$

Using this, we can rewrite the Lagrangian as:

(7)′
$$\begin{aligned} L = &\int_0^{T^*} \left\{ e^{-rt}U(c(t)) + \dot{\lambda}(t)K(t) + \lambda(t)\left[F(K(t)) - c(t) - \delta K(t)\right]\right\} dt \\ &- \mu\left[K(0) - A\right] + \vartheta K(T^*) - \lambda(T^*)K(T^*) + \lambda(0)K(0) \end{aligned}$$

(Note that the running calendar time only enters the integral part, whereas the remaining parts of the Lagrangian are related to initial and terminal constraints on the state variable.)

Now we can put up the partial derivatives (in an infinite number because time is continuous):

$$(8) \quad \frac{\partial L}{\partial K(0)} = -\mu + \lambda(0) = 0 \quad (\mu = 0 \; if \; K(0) < A)$$

$$(9) \quad \frac{\partial L}{\partial K(T^*)} = \vartheta - \lambda(T^*) = 0 \quad (\vartheta = 0 \; if \; K(T^*) > 0)$$

$$(10) \quad \frac{\partial L}{\partial c} = e^{-rt}U'(c(t)) - \lambda(t) = 0 \text{ for any } t \in \left[0, T^*\right]$$

$$(11) \quad \frac{\partial L}{\partial K} = \dot{\lambda}(t) + \lambda(t)\left[F'(K(t)) - \delta\right] = 0 \text{ for any } t \in \left(0, T^*\right)$$

Before we show how this solution can be derived from the so-called maximum principle, let us just restate some of the previous results, now in continuous time: If we return to our discrete version with, a period length equal to one, we had:

$$(10)' \quad \beta^t U'(c_t) = \lambda_t \quad \rightarrow \quad e^{-rt}U'(c(t)) = \lambda(t)$$

$$(11)' \quad \lambda_{t+1}\left[1 + F'(K_t) - \delta\right] = \lambda_t \Leftrightarrow \frac{\lambda_t - \lambda_{t+1}}{\lambda_{t+1}} = F'(K_t) - \delta \rightarrow [-\frac{\dot{\lambda}(t)}{\lambda(t)}] = F'(K(t)) - \delta$$

where the real rate of return on capital per unit of time is $F'(K(t)) - \delta := \theta(t)$. The relative rate of decline in the present discounted shadow price, $-\frac{\dot{\lambda}(t)}{\lambda(t)}$, which we have seen can be interpreted as a required rate of return on non-consumption, is equal the real net rate of return on capital, along an optimal path. This is also seen by differentiating (10) with respect to time, which yields a famous formula, first derived by Ramsey in 1928, when we define (the absolute value of) Frisch's flexibility of marginal utility, as $\hat{\omega}(c) = -\frac{c}{U'(c)}U''(c) > 0$:[4]

[4] Under certain assumptions it can be shown that the higher (smaller) is the absolute value of this Frisch flexibility, the less (better) are the intertemporal substitution possibilities; i.e., the more (less) curved are the indifference surfaces.

$$(12)\quad -r\lambda(t)+e^{-rt}U''(c(t))\cdot c(t)\cdot\frac{\dot{c}(t)}{c(t)}=\dot{\lambda}(t)\Leftrightarrow-\frac{\dot{\lambda}(t)}{\lambda(t)}=r+\hat{\omega}(c(t))\cdot\frac{\dot{c}(t)}{c(t)}:=\rho(t)$$

The RHS of the last term in (12), which is also a measure of *the social rate of discount*, is the continuous counterpart to the required rate of return from non-consumption, being equal to the sum of the instantaneous utility discount rate (a subjective element) or the rate of pure impatience, r, and the product of the absolute value of the flexibility of the marginal utility of consumption and the growth rate in consumption. (Whereas the felicity rate captures pure impatience, the second one is related to how a change in consumption is evaluated over time; a high value of $\hat{\omega}$ indicates that the benefit of additional consumption is declining rapidly.) A high discount rate or a required rate of return per unit of time in units of the macro commodity, $\rho(t)$, means that we are willing to sacrifice less today so as to get more consumption in the future. Along an optimal path the required rate of return is balanced against what can be obtained through giving up consumption today for capital accumulation, the social rate of return on investment. Along an optimal path we have:

$$(12)'\quad \rho(t):=r+\hat{\omega}(c(t))\cdot\frac{\dot{c}(t)}{c(t)}=-\frac{\dot{\lambda}(t)}{\lambda(t)}=F'(K(t)-\delta:=\theta(t)$$

Optimality over time will require a balancing of non-consumption and investment which is "undertaken" by the rate of change in the shadow price so as to equalize ρ and θ. (This common value will, under certain assumptions, be equal to the competitive rate of interest, in units of consumption.)

Let us then see how this solution can be obtained from the Maximum Principle. We define what is called the present value Hamiltonian function, along similar lines as we did when introducing the Lagrangian:

$$(13)\quad H(K,c,\lambda,t)=e^{-rt}U(c)+\lambda\left[F(K)-\delta K-c\right]$$

The Maximum principle tells us now that an optimal solution is found as choosing a control variable at each instant of time, here the consumption flow, which maximizes the Hamiltonian, when there exists an adjoint or costate variable $\lambda(t)$, that moves over time as in (11).

Hence, our (interior) optimum is found as:

$$(14)\begin{cases} H_c = e^{-rt}U'(c) - \lambda = 0 \quad \forall t \in \left[0, T^*\right] \\ \dot{\lambda}(t) = -H_K = -\lambda\left[F'(K) - \delta\right] \quad \forall t \in \left[0, T^*\right] \\ \lambda(T^*) \text{ determined so that } K(T^*) = 0 \end{cases}$$

with $\dot{K}(t) = F(K(t)) - c(t) - \delta K(t)$ satisfied as well. These conditions, which are necessary, coincide with the ones we have derived above.

Remark 4:
We can have different types of end-point constraints on the state variable, depending on the problem. With a fixed horizon, we have required that $K(T^*) = 0$, meaning that we will eat up everything before the end of the planning period. Above, see (9), we found that $\lambda(T^*) = \vartheta \geq 0$, which says that the costate variable λ now is to be determined at the end of the planning period. This condition can then be expressed as $\lambda(T^*)K(T^*) = 0$. Another terminal state constraint is to impose no condition on $K(T^*)$; we then say that the state variable at this point in time is free as we impose no constraint on how much capital we will end up with at T^*. In that case we will have $\lambda(T^*) = 0$. We can also operate with an end-point of the planning period that itself is part of the problem ("when is it optimal to stop?"), or we can have an infinite horizon. The terminal conditions imposed on the state variables will therefore require different conditions on the associated costate variables at the end-point. These conditions are called transversality conditions which we come back to in the lectures. (Note that these conditions are rather trivial with a fixed horizon, but much more complicated with infinite horizon. If we let the horizon in our problem go to infinity, we have for the end-point constraint $\lim_{t\to\infty} K(t) \geq 0$, a set of transversality conditions being $\lim_{t\to\infty} e^{-rt}\lambda(t) \geq 0$ and $\lim_{t\to\infty} e^{-rt}\lambda(t)K(t) = 0$, as a part of the solution. These conditions will rule out leaving valuable assets unused as $T \to \infty$.[5]

Remark 5:

It can be shown that with an optimal solution as in (14), and given by $\left\{c^*(t), K^*(t)\right\}_{t=0}^{t=T^*}$, with $K^*(0) = A,$ and $K^*(T^*) = 0$, then we can define a *value function*, written as

[5] These conditions will imply the so-called "No Ponzi Game" condition known from macro, and will rule out speculations and bubbles.

(15) $$V(A) = \int_0^{T^*} e^{-rt} U(c^*(t))dt$$

It is easy to demonstrate that the increment in the maximal present value of discounted utility of a marginal increase in initial capital equipment, is given by the value of the shadow price or costate variable at $t = 0$; i.e., $\lambda(0)$; hence we have:

(16) $$V'(A) = \lambda(0)$$

(Or a bit more general, if we were given (exogenously and unexpectedly) more capital equipment at some point in time, τ, it can be demonstrated that $V'(K_\tau) = \lambda(\tau)$. If we had imposed some more restrictive terminal capital requirement, say we want to transfer more capital for the period after T – whatever that means – from $K(T^*) = 0$ to $K(T^*) \geq K_T$, then we have $V'(K_T) = -\lambda(T^*)$, which is the total welfare loss of leaving more capital to the future.)

An example

Suppose we now have a simple linear production function, a logarithmic utility function and no depreciation of capital equipment. Hence we have:
$F(K) = bK$, $U(c) = \ln c$, $\delta = 0$ and a fixed planning period $\left[0, T\right]$.

The planning problem is then:

$$Max \int_0^T e^{-rt} \ln c(t)dt$$

Subject to

$$\dot{K}(t) = bK(t) - c(t); K(0) = k > 0, K(T) = 0; \textit{with } T \textit{ fixed}$$

(The model can also illustrate a pure consumption-wealth management problem of an individual, with K as wealth, and b the market rate of interest. Or it can illustrate a simple model illustrating extraction of a non-renewable resource, with K as the remaining reserve of resource, when putting $b = 0$.)

The present value Hamiltonian is $H(K, c, \lambda, t) = e^{-rt} \ln c + \lambda\left[bK - c\right]$, and according to our previous results, an optimal solution must obey:

$$H_c = \frac{e^{-rt}}{c} - \lambda = 0 \Rightarrow c(t) = \frac{e^{-rt}}{\lambda(t)}$$

$$\dot{\lambda}(t) = -H_K = -\lambda b \Rightarrow \lambda(t) = \lambda(0)e^{-bt}$$

$\lambda(T)$, and hence, $\lambda(0)$, is determined from $K(0) = k,$ and $K(T) = 0$

Inserting for c into the differential equation for the state variable, we get:

$\dot{K}(t) - bK(t) = -\frac{e^{(b-r)t}}{\lambda(0)}$, with the following solution:

$$K(t) = Be^{bt} + \frac{1}{r\lambda(0)}e^{(b-r)t}$$

Now, because we have $K(0) = k$ and $K(T) = 0$, we can determine the two constants B and $\lambda(0)$, from:

$$K(T) = 0 = Be^{bT} + \frac{1}{r\lambda(0)}e^{(b-r)T}$$

$$K(0) = k = B + \frac{1}{r\lambda(0)} \Rightarrow B = k - \frac{1}{r\lambda(0)}$$

Inserting for B in the end-point constraint, we get:

$$K(T) = 0 = e^{bT}\left[k - \frac{1}{r\lambda(0)}\right] + \frac{1}{r\lambda(0)}e^{(b-r)T} = e^{bT}\left[k - \frac{1}{r\lambda(0)}(1 - e^{-rT})\right]$$

from which we get:

$$\lambda(0) = \frac{1 - e^{-rT}}{rk}$$

Then:

$$\lambda(t) = \frac{1 - e^{-rT}}{rk}e^{-bt} \text{ and } c(t) = \frac{rk}{1 - e^{-rT}}e^{(b-r)t} := c(0)e^{(b-r)t}.$$

Finally, we get:

$$K(t) = Be^{bt} + \frac{1}{r\lambda(0)}e^{(b-r)t} = (k - \frac{1}{r\lambda(0)})e^{bt} + \frac{1}{r\lambda(0)}e^{(b-r)t} = ke^{bt}\left[1 - \frac{1 - e^{-rt}}{1 - e^{-rT}}\right]$$

Note that $-\frac{\dot{\lambda}(t)}{\lambda(t)} = b = r + \hat{\omega}\frac{\dot{c}(t)}{c(t)} = r + 1 \cdot (b - r)$, as we now have a constant growth rate of consumption; $\frac{\dot{c}}{c} = b - r$, and as $U(c) = \ln c$, $\hat{\omega} = -\frac{C}{U'(c)}U''(c) = -\frac{c}{\frac{1}{c}}(-\frac{1}{c^2}) = 1$.

Thus all the time functions for the interesting variables are then determined.

Appendix – Dorfman's derivation of the maximum principle

In a small note published in 1969, Robert Dorfman outlined the basic principles behind control theory, without entering into too heavy formalism.[6] We will give a brief survey of his approach in this appendix.

The decision unit under consideration; a firm, consumer or the a government, has, at any point in time, some instruments to be used so as to maximize the sum of future benefits, when the choice of the instruments will affect the future states or possibilities facing the decision unit. We can imagine a Ramsey economy, where the instantaneous flow utility is in general a function of the instrument or control variable x (say consumption, but might be a vector, as well), the state variable k (say capital equipment used as input in a production function, might also be a vector) and time t (to capture, say, discounting); in the sense that utility during a short period of time dt is given by $U(k,x,t)dt$. The decision unit has preferences over the integral or "sum" of all such benefits over a long period of time, $\left[0,T\right]$, as given here by

$$W(k_0,\vec{x}) = \int_0^T U(k(t),x(t),t)dt.$$

(The notation $\vec{x}$ means the sequence of future choices of instruments.) The "value" function $W(k_0,\vec{x})$ is the total value today ($t=0$), if we start out with a capital stock k_0 initially, and make decisions continuously from today until the end of the planning period, as given by the path $\vec{x} = \left\{x(\cdot)\right\}_{t=0}^{t=T}$.

In addition, there is a relationship between the choice made at some point in time and the *change* in the state variable per unit of time, along the same line as between consumption and capital accumulation above. Hence we impose the state equation showing the motion, dynamics or change per unit of time in the state variable as a function of the state variable itself, the choice (control) variable and time. Hence we impose the following constraints or state equations on a short interval of time, saying that the change in the state variable during this short time period, dk, is a function of the rate of change per unit of time $f(k,x,t)$, an intensity, multiplied by the length of the time period; cf. footnote 1 above:

[6] See R.Dorfman (1969), An Economic Interpretation of Optimal Control Theory, *American Economic Review* 59 (5), pp. 817 – 831, with some corrections provided in *American Economic Review,* 60 (3), p.524.

(1) $dk = f(k(t), x(t), t)dt \Rightarrow \dot{k}(t) = \frac{dk(t)}{dt} = f(k(t), x(t), t)$

The instantaneous choice of an instrument will have two effects: One is the direct effect on the current benefit, as given by $U(k, x, t)dt$, and another one related to the rate of change in the state variable in (1) and hence future possibilities or options.
The problem as we have seen it above is now to choose a decision path $\vec{x}$ so as to maximize the value of W, when taking into account the impact of current decisions on the rate of change in the state variable, and hence the future value of the state variable. This procedure can be followed from any arbitrary point in time, with some given initial state, as given, say at *t*, by k_t. On following, from *t* and onwards, the choice of instruments, $\vec{x}$, so far quite arbitrary (but feasible; i.e. within the set of admissible controls), the value of the objective is: $W(k_t, \vec{x}) = \int_t^T U(k(\theta), x(\theta), \theta)d\theta$.

Then we make the following trick, by breaking up this value in two parts; one of a very, very short interval of time $\left[t, t+\Delta\right]$, on which the decision taken is (the flow) x_t, along with a state variable k_t, and another one related to the value W on the interval $\left[t+\Delta, T\right]$, with an initial state $k_{t+\Delta}$. Hence we can now write:

(2) $W(k_t, \vec{x}, t) = U(k_t, x_t, t)\Delta + \int_{t+\Delta}^T U(k(\theta), \vec{x}, \theta)d\theta = U(k_t, x_t, t)\Delta + W(k_{t+\Delta,}, \vec{x}, t+\Delta)$

These "value" functions are related to an arbitrary decision path. If the decision maker now can put up the best choice of the instruments from *t* and onwards, we can define a true value function as: $V(k_t, t) = Max_{\vec{x}} W(_t, \vec{x}, t)$
Suppose now we follow some policy, x_t, not necessarily the best, for a short period of time $\left[t, t+\Delta\right]$, but from $t+\Delta$ and onwards, the best policy is followed. From (2), we then have:

(2)$'$ $W(k_t, \vec{x}, t) = U(k_t, x_t, t)\Delta + V(k_{t+\Delta,}, t+\Delta) := v(k_t, x_t, t)$

The value of pursuing or following this policy is of course the short-term benefit over the very short period, plus the maximal value following the best policy, from then on, given the corresponding initial state at the initial date $t+\Delta$. Given this way of formulating our original problem, we have turned it into a standard problem of finding the value of the scalar x_t, when taking into account the corresponding change in the state variable, which maximizes $v(k_t, x_t, t)$. Then the LHS of (2)′ is equal to $V(k_t, t) = \max_{x_t} v(k_t, x_t, t)$.

Given that the first-order condition fully characterizes an interior maximum, we then have:

$$(3) \quad \frac{\partial v(k_t, x_t, t)}{\partial x_t} = \Delta \frac{\partial U(k_t, x_t, t)}{\partial x_t} + \frac{\partial V(k_{t+\Delta,}, t+\Delta)}{\partial x_t} = 0$$

There are two problems here: First we don't yet know the function V, which does not even have x_t as an argument. How is this last problem solved? We know that the current decision will affect the subsequent initial state; hence we can write: $\frac{\partial V(k_{t+\Delta,}, t+\Delta)}{\partial x_t} = \frac{\partial V(k_{t+\Delta,}, t+\Delta)}{\partial k_{t+\Delta}} \cdot \frac{\partial k_{t+\Delta}}{\partial x_t}$. Also, because the time period $\left[t, t+\Delta\right]$ is very short, we can use the following approximation $k_{t+\Delta} = k_t + \Delta \frac{dk}{dt} = k_t + \Delta f(k_t, x_t, t)$; hence we have $\frac{\partial k_{t+\Delta}}{\partial x_t} = \Delta \frac{\partial f(k_t, x_t, t)}{\partial x_t}$. But what about $\frac{\partial V(k_{t+\Delta,}, t+\Delta)}{\partial k_{t+\Delta}}$? This is, cf. Remark 5 above, equal to the shadow value of real capital equipment, providing a measure of the welfare effect of having more capital at $t+\Delta$, which we have defined by $\lambda(t+\Delta)$. Using both types of information in (3), we have:

$$(4) \quad \frac{\partial v(k_t, x_t, t)}{\partial x_t} = \Delta \frac{\partial U(k_t, x_t, t)}{\partial x_t} + \lambda(t+\Delta) \cdot \Delta \frac{\partial f(k_t, x_t, t)}{\partial x_t} = 0$$

Cancelling Δ while using that (for a small Δ), we may write: $\lambda(t+\Delta) = \lambda(t) + \Delta \dot{\lambda}(t)$, which is used in (4) to yield:

$$(4)' \quad \frac{\partial U(k_t, x_t, t)}{\partial x_t} + [\lambda(t) + \Delta \dot{\lambda}(t)] \frac{\partial f(k_t, x_t, t)}{\partial x_t} = 0$$

Let $\Delta \downarrow 0$, we get the condition for an optimal control:

$$(5) \quad \frac{\partial U(k_t, x_t, t)}{\partial x_t} + \lambda(t) \frac{\partial f(k_t, x_t, t)}{\partial x_t} = 0$$

The current (short-term) marginal gain is balanced or traded off against the long-term effect of the current decision or the marginal long-run cost; cf., (4), (4)′, (4)′′, (10) and the first part of (14) in section 2.

Suppose that the control variable is set so as to obey (5). In that case we have:

$V(k,t) = U(k, x_t, t)\Delta + V(k(t+\Delta), t+\Delta)$. Differentiate this with respect to k, so as to find the optimal state variable, when using the definitions of the shadow prices as derived above. (Note that *k* itself is *not* a decision variable; the condition just tells us what value *k* must take in an optimal solution.)

Then we have, with our previous approximations:

$$\frac{\partial V(k,t)}{\partial k} =_{def} \lambda(t) = \Delta \frac{\partial U}{\partial k} + \frac{V(k(t+\Delta), t+\Delta)}{\partial k} = \Delta \frac{\partial U}{\partial k} + \frac{V(k(t+\Delta), t+\Delta)}{\partial k(t+\Delta)} \frac{\partial k(t+\Delta)}{\partial k}$$

$$=_{def} \Delta \frac{\partial U}{\partial k} + \lambda(t+\Delta) \frac{\partial k(t+\Delta)}{\partial k} =_{apprx} \Delta \frac{\partial U}{\partial k} + \lambda(t+\Delta) \frac{\partial}{\partial k}\left[k(t) + \Delta \dot{k}(t)\right]$$

$$=_{apprx} \Delta \frac{\partial U}{\partial k} + [\lambda(t) + \Delta \dot{\lambda}(t)]\left[1 + \Delta \frac{\partial f}{\partial k}\right] = \Delta \frac{\partial U}{\partial k} + \lambda(t) + \Delta \dot{\lambda}(t) + \lambda(t)\Delta \frac{\partial f}{\partial k} + \Delta^2 \dot{\lambda}(t) \frac{\partial f}{\partial k}$$

Cancelling out $\lambda(t)$ on both sides, divide through by Δ and then let $\Delta \downarrow 0$, we get:

$$(6) \quad -\dot{\lambda}(t) = \frac{\partial U}{\partial k} + \lambda(t) \frac{\partial f}{\partial k}$$

This is the analog to (5), (5)′, (5)′′ and (11) in section 2. The interpretation of (6), according to Dorfman is: "To a mathematician, $\dot{\lambda}$ is the rate at which the value of a unit of capital is changing. To an economist, it is the rate at which the capital is appreciating. $-\dot{\lambda}$ is therefore the rate at which a unit of capital depreciates at time *t*. Accordingly the formula asserts that when the optimal time path of capital accumulation is followed, the decrease in value of a unit of capital in a short interval of time is the sum of its contribution to profits realized during the interval and its contribution to enhancing the value of the capital at the end of the interval."

Therefore the conditions (5), (6) and (1) along with initial state and some terminal constraint on the state variable, will fully characterize an intertemporal optimum, as given by $\left\{x(t), k(t), \lambda(t); k(0)=k_0,\ a\ condition\ on\ k(T)\right\}_{t=0}^{t=T}$.

In a specific problem, cf. our example above, we can (in principle) use (5) to write $x(t)=\phi(k(t),\lambda(t),t)$, which can be inserted into the two remaining differential equations to give a system of differential equations in $(k(t),\lambda(t))$ which in principle can be solved:

$$(1)' \quad \dot{k}(t)=f(k(t),\phi(k(t),\lambda(t),t),t) := F(k(t),\lambda(t),t)$$

$$(6)' \quad -\dot{\lambda}(t)=\frac{\partial U(k(t),\phi(k(t),\lambda(t),t),t)}{\partial x}+\lambda(t)\frac{\partial f(k(t),\phi(k(t),\lambda(t),t),t)}{\partial k} := \Lambda(k(t),\lambda(t),t)$$

With given initial and terminal state constraints, $k(0)=k_0$, and, say, $k(T)=k_T\geq 0$, we should be able to determine the entire capital path, as well as the shadow price path, with endogenously determined values of $\lambda(0)$ and $\lambda(T)$, due to the initial and terminal state constraints.

Remark 6:
The conditions derived above are necessary for an optimal solution. If the functions U and f are sufficiently differentiable and jointly concave in (x,k), then these conditions are sufficient as well. We need as always be cautious for not having found a minimum when we want to find a maximum of some problem. Concavity as noted above will therefore almost guarantee that the necessary first-order conditions derived here are sufficient.