Available online at www.sciencedirect.com

SCIENCE DIRECT®

ELSEVIER

Journal of Environmental Economics and Management 50 (2005) 408–421

JOURNAL OF ENVIRONMENTAL ECONOMICS AND MANAGEMENT

www.elsevier.com/locate/jeem

How to eat a cake of unknown size: A reconsideration

Ramesh C. Kumar*

Department of Economics, University of Waterloo, 200 University Avenue West, Waterloo, Ont., Canada N2L 3G1

Received 3 March 2003
Available online 9 April 2005

Abstract

The paper revisits the issue of the optimal depletion of an exhaustible resource stock of uncertain size by recasting it in terms of the hazard function. In addition to re-establishing some of the existing results, it obtains a complete, qualitative characterization of the optimal depletion program for a fairly large array of continuous probability distributions that are likely to describe the initial stock distribution. It turns out that an important feature of the optimal program is the eventual monotonicity of the optimal extraction-cum-consumption rate, so characteristic of the certainty scenario. Additional results regarding the duration of the optimal planning horizon and comparison with the situation of perfect certainty provide further insight into the nature of the optimal depletion program for the iso-elastic utility function.

Keywords: Cake-eating; Uncertainty; Exhaustible resources; Optimal planning horizon; Hazard function

1. Introduction

In this paper, we revisit the issue of the optimal depletion of an exhaustible resource stock of uncertain size, first considered by Kemp [8]. Commonly known in the literature as the problem of cake-eating under uncertainty, it has two important aspects: the optimal planning horizon and the characterization of the optimal program. Since the resource amount to be depleted is uncertain by assumption, we may surmise that the optimal planning horizon extends sufficiently far into the future to permit the full exhaustion of whatever is ultimately discovered underground. But, should

*Fax: +1 519 725 0530.
E-mail address: rkumar@watarts.uwaterloo.ca.

doi:10.1016/j.jeem.2005.01.003

this automatically imply that the optimal planning horizon is always infinite, as has been generally assumed (e.g., [3,4,6,17,18], among others), following the example set by Kemp himself? Or, is it possible that the optimal planning horizon under some circumstances is of finite duration only, as recently shown in Kumar [13]? [1] As regards the characterization of the optimal program, Kemp demonstrated that unlike the case of perfect certainty, the optimal extraction-cum-consumption rate was not necessarily decreasing over time. Ever since, the overall picture has not changed much, except for the contributions by Gilbert [3,4] and Loury [17] immediately following Kemp, and showing that the optimal depletion rate is constant if the initial resource stock is distributed exponentially.

In what follows, we attempt to make further progress with respect to both aspects of the problem by making use of the hazard function, a popular tool in duration-data and survival analyses. In particular, we extend the definitive result in Kumar [13], which holds only in the context of a discrete probability distribution of the initial resource stock, to all continuous distributions with finite support and increasing hazard function. Recasting the problem in this manner also enables us to derive a complete, qualitative characterization of the optimal depletion program for a variety of probability distributions and utility functions, thereby allowing us to view the Gilbert and Loury result as a special case. Additional results regarding the expected optimal time horizon and comparison with the situation of perfect certainty help further characterize the optimal program for the special case of the iso-elastic utility function.

We organize the remainder of the paper as follows. In Section 2, we outline the cake-eating problem under uncertainty and review the results achieved to date. We devote Section 3 to the derivation of our results, and Section 4 to presenting the conclusions.

2. The problem

Let the random variable S represent the size of the uncertain resource stock at the start of the planning period. Denote by $f(s)$ and $F(s)$, $s \in (0, \bar{s}], \bar{s} \leqslant \infty$, the probability distribution and the cumulative probability distribution of S, respectively. Assume that S possesses a finite mean (expected value) and a finite variance. Also, let $q(t)$ stand for the planned extraction-cum-consumption rate at instant t and, $u(q(t))$, for the corresponding instantaneous utility. It is customary to assume that $u(q(t))$ is everywhere non-negative and strictly concave. It is also customary to assume that $u(0) = 0$, that is, no cake implies no utility. If $\delta > 0$ is the constant time-preference rate or the discount rate, then the cake-eating problem under uncertainty consists of specifying a contingent extraction-cum-consumption program, $\{q(t)\}_{t=t_0}^{t=\tau}$, which maximizes

$$E\left[\int_{t=t_0}^{t=\tau} e^{-\delta t} u(q(t))\, dt\right] \tag{1}$$

[1] It must be noted that Kemp was aware of the possibility. He noted, for example, "[t]he assumption that the owner of the cake is immortal is immaterial," and mentioned the possibility of the arrival of "a moment of sorrow" when consumption of the cake might drop to zero.

subject to

$$\int_{t=t_0}^{t=\tau} q(t)\,dt \leqslant S, \quad q(t) \geqslant 0, \tag{2}$$

where t_0 is the beginning of the planning horizon and τ, a random variable, denotes the uncertain terminal date at which the resource stock is fully depleted.[2] The expectation in (1) is taken over the probability distribution of τ as dictated by $f(s)$ and the planned path of $q(t)$. If $G(t)$ now stands for the cumulative probability distribution of τ, it follows that

$$G(t) = Prob(\tau \leqslant t) = Prob\left(S \leqslant \int_0^t q(r)\,dr\right) = F\left(\int_0^t q(r)\,dr\right) = F(Q(t)), \tag{3}$$

where

$$Q(t) = \int_0^t q(r)\,dr \tag{4}$$

is the cumulated planned extraction-cum-consumption to date t, and t_0 has been set to equal zero without any loss of generality. As a result, (1) assumes the form

$$\int_0^T \int_0^\tau e^{-\delta t} u(q(t))\,dt\,dG(\tau), \quad 0 < \tau \leqslant T, \tag{5}$$

where T is the least upper bound of the support of the implied probability distribution of τ and is unknown ex ante. In other words, T is the maximum time over which consumption is possibly strictly positive. Since $G(T) = F(Q(t)) = 1$ by definition, it follows that T is such that $\lim_{t\to T} Q(T) = \bar{s}$, and rewriting (1) in this manner transforms the problem of determining the optimal planning horizon into one of specifying a value for T. Clearly, T can be finite or infinite.

Unless stated otherwise, we assume in the rest of the paper that $f(s)$ is continuous and differentiable over $(0, \bar{s}]$.[3] Thus $dG(t) = dF(Q(t)) = f(Q(t))q(t)\,dt$ with $G(0) = F(Q(0)) = 0$ and $G(T) = F(Q(T)) = 1$, and the functional in (5) may be written as

$$\int_0^T f(Q(\tau))q(\tau)\left[\int_0^\tau e^{-\delta t} u(q(t))\,dt\right] d\tau. \tag{6}$$

Upon integration by parts and appropriate substitutions (6) becomes

$$\int_0^T e^{-\delta t} u(q(t))[1 - F(Q(t))]\,dt, \tag{7}$$

thereby transforming the preceding stochastic optimal control problem into a deterministic one of maximizing the functional in (7) subject to the constraints

$$\dot{Q}(t) = q(t), \quad q(t) \geqslant 0 \tag{8}$$

[2] Strictly speaking, this is true only if $u(0) = 0$. In case $u(0) > 0$, the expression in (1) must be expanded to include the certain benefits that may accrue following resource exhaustion.

[3] Strictly speaking, all we need assume is piece-wise continuity and differentiability.

and the boundary conditions

$$Q(0) = 0, \quad \lim_{t \to T} Q(t) = \bar{s} \tag{9}$$

with $T > 0$ free.[4] The terminal boundary condition, which is not always explicitly stated, is simply a restatement of the requirement that $G(T) = F(Q(T)) = 1$ and implies that the support of τ extends just sufficiently far into the future to allow for the complete exhaustion of the maximum possible resource amount in ground.

Koopmans [11,12] is generally recognized as being the first to specifically consider the issue of the optimal time horizon in the context of a cake-eating problem. While investigating the issue under perfect certainty, he observed that if survival required a minimum level of consumption, the maximum time-period over which the economy could survive—the optimal resource exhaustion time—was finite and shorter than that for $\delta = 0$ as long as $u(q(t))$ was bounded from above and $\lim_{q(t) \to \infty} u'(q(t)) = 0$. Subsequently, Dasgupta [1] and Dasgupta and Heal [2] pointed out that if $u(q(t))$ was not bounded from above and $\lim_{q(t) \to 0} u'(q(t)) = \infty$, $q(t)$ was trivially assured to be always positive so that a finite resource exhaustion time could not be optimal. At about the same time, Vousden [19] observed that if the exhaustible resource was not the only source of consumption, creating the possibility of $u(0) > 0$, a clear distinction needed to be drawn between the survival and resource exhaustion times in that the former could be considerably longer. Since then Highfill and McAsey [5] and Kumar and Naqib [14] have further expanded the scope of the Dasgupta–Heal result by proving that, provided $u(0) = 0$, the limiting behavior of $u'(q(t))$ is the sole determinant of the optimal T under perfect certainty such that optimal $T < \infty (= \infty)$ if and only if $\lim_{q(t) \to 0} u'(q(t)) < \infty (= \infty)$.[5] More recently, Kumar[13] has shown that this perfect certainty result can be fully extended to the case of resource stock uncertainty as well, provided it is represented by a discrete probability distribution admitting at most a finite number of values.

The history of the characterization of the optimal extraction-cum-consumption program or the depletion policy is even shorter. Almost at the heal of the original contribution by Kemp, Loury [17] demonstrated that optimal $q(t)$ was constant if and only if $f(s)$ was exponential. Gilbert [4] also derived a similar result based on a highly specialized, discrete $f(s)$ with unbounded support, but illustrated his finding for iso-elastic $u(q)$ and exponential $f(s)$. However, both Loury and Gilbert presumed optimal T to be infinite. While Kumar [13] has recently attempted to deal with both issues simultaneously, the suggested procedure is developed in the context of a discrete probability distribution with finite support and cannot be extended to the case of a continuous one, the focus of both the original Kemp study and the attempt here. The difficulty, in our view, lies in the differential information extraction generates in the two cases. When $f(s)$ is discrete, continuing extraction beyond the level of proven reserves reveals at infinitesimal cost (because of the infinitesimal extraction effort necessary for the purpose) the minimum amount of the resource still under ground. This transforms the problem into one of choosing a sequence of appropriate time horizons to optimally deplete the currently proven reserves in the light of the latest update of

[4] See Kemp [8, pp. 298–299]. Loury [17] also notes the equivalence of the two problems.

[5] Assuming $u(0) = 0$ eliminates the distinction between resource exhaustion time and survival time of the economy. In ours and the Dasgupta–Heal version of the cake-eating problem, therefore, T stands for the planning horizon, the resource exhaustion time, the survival time as well as for the so-called "moment of sorrow" of Kemp and the "doomsday" of Koopmans.

$f(s)$. This virtually costless generation of additional information is not possible in the case of a continuous probability distribution, for now infinitesimal extraction can at best reveal the presence underground of no more than an infinitesimal amount of the resource, thereby leaving the level of proven reserves unchanged for the next period.

In the next section, we attempt a solution of the problem by making use of the hazard function and show how some of these difficulties may be overcome.

3. Characterizing the optimal program

As the Hamiltonian associated with the planning problem of the preceding section is

$$H = u(q)\pi(Q)e^{-\delta t} + \lambda q, \tag{10}$$

the necessary conditions for an optimal program include, in addition to (8) and (9),

$$u'(q)\pi(Q)e^{-\delta t} = -\lambda, \tag{11}$$

$$\dot{\lambda} = -u(q)\pi'(Q)e^{-\delta t}, \tag{12}$$

$$\lim_{t \to T} H = 0, \tag{13}$$

where $\pi(Q) = 1 - F(Q)$ is the survival function; $\lambda \leqslant 0$ is the co-state variable; and the explicit dependence of the various functions on t has been suppressed for ease in notation.[6] Next, differentiating (11) with respect to t and substituting from (12) yield

$$\frac{\dot{q}}{q} = \frac{\phi(q)h(Q) - \delta}{\varepsilon(q)}, \tag{14}$$

$$\frac{\dot{\lambda}}{\lambda} = -\frac{u(q)}{u'(q)}h(Q), \tag{15}$$

where

$$\phi(q) = (u(q)/u'(q)) - q; \quad \varepsilon(q) = -u''(q)q/u'(q) > 0; \text{ and}$$
$$h(Q) = -\pi'(Q)/\pi(Q) = f(Q)/[1 - F(Q)] \geqslant 0.$$

Eqs. (14) and (15) are standard means of describing the optimal depletion policy, but we have written them in a somewhat unfamiliar form by utilizing three different functions to isolate the impact of different factors.[7]

The first two functions—$\phi(q)$ and $\varepsilon(q)$—concern different aspects of the utility function. It is easy to check that $\phi(q) \geqslant 0$ and $\phi'(q) = -u(q)u''(q)/[u'(q)]^2 > 0$ for $q \geqslant 0$, reflecting the strict concavity of $u(q)$. $\varepsilon(q)$ is a measure of the degree of risk aversion implied by the concavity of $u(q)$ and is always positive.

[6] The difference between (13) and the transversality condition stipulated in Kemp [8] is entirely due to our contention that T is endogenous and may not be pre-specified in the case of an uncertain stock.

[7] For example, (14) is exactly the same as Eq. (18) in Loury [17, p. 626].

The third function, $h(Q)$, called the hazard function, is an alternative means of describing the uncertainty aspects of the problem. In stricter probability terms, it is an instantaneous failure rate used to specify the conditional probability that the spell of the event described by a random variable will not last beyond when it takes a specific value, given that it has lasted till then.[8] In the present context, the event of interest is the exhaustion of the uncertain resource stock S so that $h(Q(t))\Delta Q(t)$ may be interpreted as the conditional probability that S will be fully exhausted once cumulated extraction reaches $Q(t)$, given that it is initially at least as large as $Q(t)$; or, equivalently, the conditional probability that S will be fully exhausted at instant t given that positive extraction has lasted until then.[9]

As regards the properties of $h(Q)$, no restrictions other than non-negativity may be generally imposed a priori. In the context of exhaustible natural resources, however, we may surmise that it is monotonic and increasing in Q if $\bar{s}<\infty$. While a constant hazard rate implies that the conditional probability that the resource stock will be exhausted shortly remains unchanged regardless of how large an amount has been already extracted—clearly unrealistic in the face of an ultimately finite stock—a declining hazard function suggests an even more unrealistic scenario. The said probability actually declines with cumulated extraction. In what follows, therefore, we associate a finite $\bar{s}$ with only a rising hazard function.[10] However, we make no such stipulations if $\bar{s}=\infty$.

We begin our characterization of the optimal depletion program with the case of $\bar{s}<\infty$. Under the assumption that $h'(Q)>0$ always, consider the extraction decision for period t. A rising hazard function, as explained above, implies that the likelihood that the uncertain resource stock, having generated positive extraction to date, will be fully exhausted at the close of the period is greater than in the preceding period. Given that this is the only initial information that gets updated between periods, an increasing hazard function is a signal that the uncertain resource stock may be smaller than previously expected. Other things remaining equal, reducing the extraction rate would appear to be the rational economic response. As t can stand for any time-period, a continuously declining extraction rate until exhaustion would appear to be the optimal depletion policy. Moreover, the optimal extraction rate at exhaustion would be zero if the hazard were to rise in unbounded fashion.

But, how long does it take to exhaustion? Intuitively, continuing positive extraction and the consequent extension of the time-horizon is desirable as long as expected benefits—$u'(q)\pi(Q)e^{-\delta t}$—from doing so are positive. Since marginal utility of extraction rises along the

---

[8] For a non-negative, continuous random variable X, the hazard function is defined as

$$h(x)=\lim_{\Delta x\to 0}\frac{P(x\leqslant X<x+\Delta x|X\geqslant x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{P(x\leqslant X<x+\Delta x)/P(X\geqslant x)}{\Delta x}=\frac{f(x)}{1-F(x)}=-\frac{\pi'(x)}{\pi(x)},$$

where $\pi(x)$ is the survival rate for X. See [7]. Kiefer [9] and Lancaster [15] are excellent sources for economic and econometric applications.

[9] Since $G(t)=F(Q(t))$, the hazard function for exhaustion time, τ, may be written as

$$h(t)=\lim_{\Delta t\to 0}\frac{P(t\leqslant\tau<t+\Delta t|\tau\geqslant t)}{\Delta t}=\frac{dG(t)/dt}{1-G(t)}=\frac{f(Q(t))q(t)}{1-F(Q(t))}=-\frac{\pi'(Q(t))q(t)}{\pi(Q(t))}.$$

.

[10] The fact that an increasing hazard rate is normally associated with survival models involving natural aging or wear provides further support for the stipulated characterization. See [10,16].

optimal program, expected benefits are not likely to become zero in finite time if marginal utility is not bounded from above. By the same token, a finite time horizon may become possible if marginal utility at zero is finite. Proposition 1 below formally confirms our intuition on these scores.

Proposition 1. *Provided* $u(0) = 0$, $\bar{s} < \infty; h'(Q) > 0$ *and* $\lim_{Q \to \bar{s}} h(Q) = \infty$, *optimal* $T = (<)\infty$ *if and only if* $u'(0) = (<)\infty$, *and optimal q continuously declines to zero.*

Proof. See appendix.[11] □

The general result of Proposition 1 is easily adopted to the more specific case of the iso-elastic utility function, leading to a more definitive characterization of the optimal depletion policy.

Corollary 1. *If* $\bar{s} < \infty$, $u(q) = q^\eta$, $0 < \eta < 1$, *and* $h'(Q) > 0$ *such that* $\lim_{Q \to \bar{s}} h(Q) = \infty$, *optimal q asymptotically declines to zero as described by the equation*: $q(t) = q(0)e^{-\frac{\delta}{1-\eta}t}\pi(Q(t))^{-\frac{1}{\eta}}$.

Proof. Given that $\lim_{q \to 0} u'(q) = \eta q^{-(1-\eta)} = \infty$, optimal $T = \infty$. The rest follows from adapting Eqs. (14) and (15) to the specific case and solving them. □

We turn next to the case when $\bar{s} = \infty$. Intuition suggests that $T < \infty$ cannot be optimal, for one could consistently extract infinite amounts of the resource at each and every instant for the duration of T and yet not exhaust the initial stock. Proposition 2 below formally confirms our conclusion under minimal restrictions on the utility function.

Proposition 2. *If* $u(0) = 0$, $\lim_{q \to \infty} u'(q) = 0$ *and a solution to the cake-eating problem of the preceding section exists*, $\bar{s} = \infty$ *implies optimal* $T = \infty$.

Proof. See appendix. □

Knowing in advance that optimal $T = \infty$ is of immense help in characterizing the optimal depletion policy, for it helps specify the optimal terminal extraction rate in terms of the limiting behavior of the hazard function. To see this more clearly, combine (11) and (12) to yield

$$\lim_{t \to \infty} u'(q(t)) = \lim_{t \to \infty} \frac{-\dot{\lambda}(t)}{\pi'(Q(t))q(t)e^{-\delta t} - \delta\pi(Q(t))e^{-\delta t}} = \lim_{t \to \infty} \frac{u(q(t))}{q(t) + \delta/h(Q(t))} \tag{16}$$

or, equivalently,

$$\lim_{t \to \infty} \phi(q(t)) = \delta/h(\bar{s}). \tag{17}$$

Given that both $u'(q)$ and $\phi(q)$ are continuous and monotonic in q, either of (16) and (17) may be used to determine uniquely optimal terminal q.

Combining this information with our intuition regarding how the hazard function impacts on optimal extraction as stated in the context of Proposition 1 leads easily to the specification of the optimal depletion policy for different characterizations of the hazard function in the manner stated in the following result.

Proposition 3. *If* $u(0) = 0$, $\lim_{q \to \infty} u'(q) = 0$, *and* $\bar{s} = \infty$, (i) $h'(Q) > 0$ *as Q becomes sufficiently large implies that, except possibly for a finite phase at the start, optimal q declines continuously to*

[11] Unless a proof is extremely short, it is relegated to the appendix in the interest of exposition.

zero (a finite positive amount) accordingly as $\lim_{Q\to\infty} h(Q) = \infty(<\infty)$. (ii) $h'(Q)<0$ *as Q becomes sufficiently large implies that, except possibly for a finite phase at the start, optimal q rises continuously in unbounded fashion (to a finite positive amount) accordingly as* $\lim_{Q\to\infty} h(Q) = 0(<\infty)$.

Proof. See appendix. □

Just as in the case of $\bar{s}<\infty$, a more definitive characterization of the optimal depletion policy can be obtained for the iso-elastic utility function, $u(q) = q^\eta$, $0<\eta<1$, while $f(s)$ remains fairly general.

Corollary 2. *If* $u(q) = q^\eta$, $0<\eta<1$, *and* $\bar{s} = \infty$, (i) *the optimal depletion policy is described by* $q(t) = q(0)e^{-\frac{\delta}{1-\eta}t}\pi(Q(t))^{-\frac{1}{\eta}}$; (ii) *except possibly for a finite phase at the start, optimal q declines continuously to zero (a finite positive amount) if* $h'(Q)>0$ *for Q sufficiently large and* $\lim_{Q\to\infty} h(Q) = \infty(<\infty)$; (iii) *except possibly for a finite phase at the start, optimal q rises continuously in unbounded fashion (to a finite positive amount) if* $h'(Q)<0$ *for Q sufficiently large and* $\lim_{Q\to\infty} h(Q) = 0(<\infty)$.

Proof. Follows directly from adapting Eqs. (14)–(16) to the special case of the iso-elastic utility function.[12] □

Corollary 2 encompasses Gilbert's [3,4] attempt at characterizing the optimal program for the iso-elastic utility function. Even though Gilbert considered only a discrete probability distribution, its description was essentially equivalent to assuming that the shape of the associated hazard function, at least in its continuous analog, was the same as that stipulated in the statement of the corollary.[13]

While Proposition 3 does not directly cover the case of the constant hazard rate implied by the exponential distribution, our method is general enough to permit a full analysis of this special case. In fact, the additional information that the underlying probability distribution is exponential enables us to obtain a somewhat more general result than that implied by Proposition 3 above.

Proposition 4. *If* $u(0) = 0$ *and* $\lim_{q\to\infty} u'(q) = 0$, $q(t) = \phi^{-1}(\delta/\gamma)$ *forever is the optimal depletion program if and only if* $f(s)$ *is exponential with the hazard rate of* γ.

Proof. See appendix. □

Proposition 4 reproduces one of Loury's results bearing on the issue of the present paper. Moreover, if we adapt Proposition 4 to analyze the even more specialized case of the iso-elastic utility function combined with the constant hazard rate of the exponential distribution, we notice at once that the constant optimal depletion rate is given by $\bar{q} = \phi^{-1}(\delta/\gamma) = \delta\eta(1-\eta)^{-1}\gamma^{-1}$ as first derived by Gilbert.

In continuing to explore further the special case of the iso-elastic utility function and exponential distribution, we note that Proposition 4 also implies

Corollary 3. *If* $u(q) = q^\eta$, $0<\eta<1$, *and* $f(s) = \gamma e^{-\gamma s}$, $s\geqslant 0$, $\infty>\gamma>0$, $E(\tau) = (1-\eta)(\delta\eta)^{-1}$ *and* $V(\tau) = [(1-\eta)/\delta\eta]^2$.

[12] The limiting value of q is given by: $\lim_{t\to\infty} q(t) = \frac{\delta\eta}{1-\eta}\lim_{t\to\infty}(h(Q(t))^{-1} = \frac{\delta\eta}{1-\eta}\frac{1}{h(\bar{s})}$.

[13] Part (ii) of the corollary. Gilbert [4, pp. 51–52].

Proof. Follows from the straightforward computation of $E(\tau)$ and $V(\tau)$ in the light of $\bar{q} = \phi^{-1}(\delta/\gamma) = \delta\eta(1-\eta)^{-1}\gamma^{-1}$. □

Hence, both the expected value and the variance of the optimal planning horizon are independent of the expected value of the uncertain stock. Though apparently counter-intuitive, the result is easily explained. Since, the constant optimal extraction-cum-consumption rate, as specified above, is directly proportional to the expected value, $1/\gamma$, of the initial resource stock, the expected optimal exhaustion time is simply the inverse of the constant of proportionality, $\delta\eta/(1-\eta)$. In other words, if the distribution of the uncertain resource stock is exponential, a larger expected stock at the start simply translates into a proportionately less conservative depletion of the resource stock. The independence of the variance, however, is merely a consequence of the special nature of the exponential distribution.

It may also be of some interest to compare the nature of the optimal depletion program as described by Propositions 1–4 with that under certainty. For iso-elastic $u(q)$ and exponential $f(s)$ Gilbert [4] has shown that although the optimal extraction policy under uncertainty is initially more conservative than that aimed at extracting $E(S)$, it eventually becomes less so. As the use of the hazard function allows us to provide a more definitive description of the optimal depletion policy under iso-elastic utility (Corollaries 1 and 2), we can considerably expand the scope of the Gilbert result.

Proposition 5. *If* $u(q) = q^{\eta}$, $0<\eta<1$, *and* $h(Q)$ *is monotonic in* Q, *the optimal depletion policy under uncertainty is more conservative than that aimed at depleting* $E(S)$ *under certainty for at most a finite initial phase.*[14]

Proof. See appendix. □

4. Conclusions

Above we have reconsidered the problem of optimal depletion of an exhaustible resource stock of an uncertain size in its two aspects—the duration of the optimal planning horizon and the characterization of the optimal depletion program—by utilizing the analytical tool of the hazard function.

With respect to the first aspect, we have shown that if the uncertain stock is ultimately finite so that the associated hazard rate rises in unbounded fashion, the optimal planning horizon is finite (infinite) accordingly as the marginal utility of extraction-cum-consumption at zero is finite (infinite).This extends the scope of the earlier result due to Kumar [13] to a fairly broad class of continuous distributions of the initial stock. If, however, the size of the uncertain stock is unbounded, we have shown that the optimal planning horizon is infinite, regardless of the underlying probability distribution.

The use of the hazard function has also allowed us to contribute with respect to the second aspect. In addition to furnishing alternative proofs of the existing results with regard to the specialized cases of the iso-elastic utility function and/or exponential distribution, we have been able to obtain a reasonably complete, qualitative characterization of the optimal depletion

[14] The original Gilbert result—[4], p. 52—may also be derived following exactly the same procedure.

program for a fairly large class of probability distributions that are likely to characterize the initial resource stock. More specifically, we have demonstrated that, regardless of the ultimate size of the uncertain stock, the optimal extraction-cum-consumption declines continuously, provided the underlying resource stock distribution possesses a continuously increasing hazard function. If, however, the underlying stock distribution were to possess a hazard function that continuously declines (remains constant) as cumulated extraction becomes sufficiently large—and these are realistic possibilities only in the case of an ultimately unbounded resource stock—the optimal extraction-cum-consumption rate continuously increases (remains constant), except for a finite phase at the start. In short, the optimal extraction-cum-consumption rate over time generally moves in monotonic fashion in the direction opposite to that of the hazard rate.

Finally, we have presented some results that help characterize further the optimal depletion program for the specialized case of iso-elastic utility. Of particular interest in this context is the demonstration that as long as hazard is monotonic, the optimal extraction-cum-consumption rate is generally more conservative than that under certainty for at most a finite initial phase.

Acknowledgements

The comments and suggestions from the co-editor and anonymous referees have been very helpful in the writing of the present version. This is gratefully acknowledged with the usual disclaimer regarding the errors that remain.

Appendix

Proof of Proposition 1. We make use of the phase diagram of q as implied by Eq. (14). Given that $u(0) = 0$ implies $\phi(0) = 0$ and that $h'(Q) > 0$ with $\lim_{Q \to \bar{s}} h(Q) = \infty$, (14) ensures that the $\dot{q} = 0$ contour, denoted by $\bar{q}(Q)$, exists such that

$$\frac{d\bar{q}(Q)}{dQ} = -\frac{\phi(\bar{q})}{\phi'(\bar{q})} \frac{h'(Q)}{h(Q)} < 0, \quad 0 \leqslant Q \leqslant \bar{s} \tag{A.1}$$

and $\bar{q}(\bar{s}) = 0$. Thus, as shown in Fig. 1, there may exist at most three possible characterizations of the optimal depletion program: optimal q rises continuously until exhaustion; optimal q rises continuously until exhaustion, except for an initial finite phase of continuous decline; and, optimal q declines continuously until exhaustion.

The first two scenarios must of necessity imply optimal $T < \infty$ and optimal $q(T) > 0$. The third, on the other hand, suggests optimal $q(T) = 0$ but says nothing about the magnitude of T. However, as $u'(q) > 0$ and $\pi(Q(T)) = 0$ by definition, it follows from (11) that $\lambda(T) = 0$ as well. Consequently, (11) and (12) together yield

$$u'(q(T)) = \lim_{t \to T} \frac{-\dot{\lambda}(t)}{\pi'(Q(t))q(t)e^{-\delta t} - \delta\pi(Q(t))e^{-\delta t}} = \lim_{t \to T} \frac{u(q(t))}{q(t) + \delta/h(Q(t))} = \frac{u(q(T))}{q(T)} \tag{A.2}$$

because $\lim_{t \to T} h(Q(t)) = h(\bar{s}) = \infty$. It is easy to see that (A.2) is equivalent to the requirement that $\phi(q(T)) = 0$, which, given our assumptions, can be satisfied if only if $q(T) = 0$. Moreover, it is

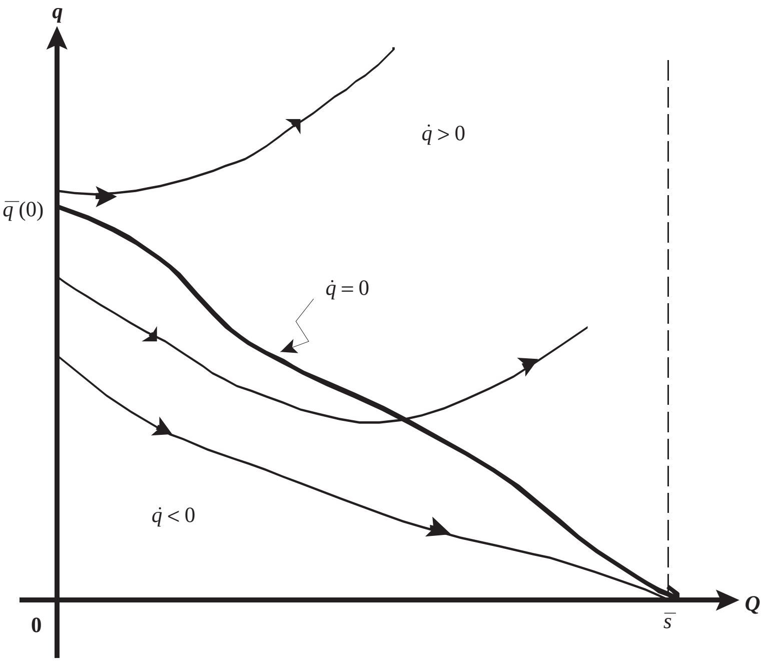

Fig. 1. Optimal program when $\bar{s}<\infty$.

easy to verify that terminal values of zero for both λ and q are also consistent with the transversality condition (13), leading to the conclusion that a unique optimal depletion policy exists and is characterized by the third possibility alone.

As regards the assertion regarding the optimal planning horizon, it suffices to show that optimal $T=\infty(<\infty)$ accordingly as $u'(0)=\infty(<\infty)$. First, upon multiplying both sides of (14) by $\varepsilon(q)$ and integrating, we obtain

$$u'(q(T)) = u'(0) = u'(q(0))e^{\int_0^T [\delta-\phi(q(t))h(Q(t))]\,dt}. \tag{A.3}$$

Optimal $\dot{q}(t)<0$ ensures that the expression within the square brackets in (A.3) is always positive so that the right-hand side of (A.3) increases with T. In view of the uniqueness of the optimal $0<u'(q(0))<\infty$, it immediately follows that optimal $T=\infty$ whenever $u'(0)=\infty$.

Next, let $u'(0)<\infty$ and suppose, if possible, that optimal $T=\infty$. Since $u''(q(t))<0$ for all q and optimal $\dot{q}(t)<0$, it follows that $u'(q(t))$, as a function of t, must approach its maximum value in an asymptotic manner only. As optimal $q(T)=0$, this must imply $\lim_{t\to\infty} u''(q(t)) = u''(0) = 0$, a contradiction because $u(q)$ is strictly concave. □

Proof of Proposition 2. Suppose, if possible, that optimal $T<\infty$. It immediately follows that optimal q must become infinite at T. Also, as $\pi(Q(T))$ must, of necessity, equal zero, (11) demands that $\lambda(T)=0$ as well. But, with $u(q)$ strictly concave, (11) and (12) together imply

$$\frac{\dot{\lambda}}{\lambda} = \frac{u(q)}{u'(q)}\frac{\pi'(Q)}{\pi(Q)} = h(Q)q - \psi(t) < 0, \tag{A.4}$$

where $0 \leqslant \psi(t) < \infty$ is a suitably defined slack function. Next, solving for $\lambda(t)$ and substituting back in (11) yields

$$u'(q(t)) = -\lambda(0)e^{\int_0^t (\delta - \psi(r))\,dr}. \tag{A.5}$$

If we now assume that $\lim_{q\to\infty} u'(q) = 0$, (A.5) presents a contradiction when $t \to T$, for, in the limit, the right-hand side remains a finite positive number. □

Proof of Proposition 3. Proceeding in the manner of Proposition 1, we note that (i)–(iv) below are sufficient for establishing the proposition. (i) The $\dot{q} = 0$ contour in the phase portrait (Fig. 2) of Eq. (14) is negatively (positively) sloped accordingly as $h'(Q) > 0 (< 0)$; (ii) the terminal q along the $\dot{q} = 0$ contour and the optimal terminal q as specified by (17) are always exactly the same; (iii) optimal terminal q is zero/finite/infinite accordingly as $\lim_{Q\to\infty} h(Q) = h(\bar{s}) = \infty / < \infty / = 0$; (iv) the uniqueness of the optimal terminal q implies that the optimal depletion policy is unique. □

Proof of Proposition 4. If $\gamma > 0$ is the constant hazard rate, (14) becomes

$$\frac{\dot{q}}{q} = \frac{\phi(q)\gamma - \delta}{\varepsilon(q)}, \tag{A.6}$$

so that the resultant $\dot{q} = 0$ contour is a horizontal line in the non-negative orthant of the (Q, q) plane and is defined by $\bar{q}(Q) = \phi^{-1}(\delta/\gamma)$. Once again, there exist at most three possible characterizations of the optimal program: optimal q rises continuously; optimal q declines continuously; or optimal q stays constant forever. However, since (17) now demands that optimal terminal $q = \bar{q} = \phi^{-1}(\delta/\gamma)$, the options of continuously rising or declining q as optimal programs are ruled out. As a constant q forever also satisfies the transversality condition, we may conclude that $q = \bar{q} = \phi^{-1}(\delta/\gamma)$ forever is the unique optimal program.

It is also quite easy to see that the converse result holds as well. If $q = \bar{q}$ forever were the optimal program, (14) would imply that $h(Q) = \delta\bar{q}/\phi(\bar{q})$ is a constant. This is sufficient to infer that the underlying $f(s)$ must be exponential with a hazard rate of $\gamma = \delta\bar{q}/\phi(\bar{q})$. □

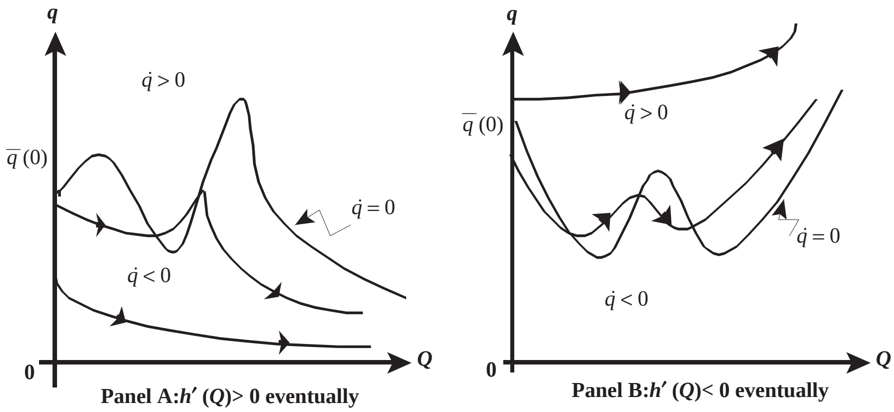

Fig. 2. Optimal program when $\bar{s} = \infty$.

Proof of Proposition 5. We consider first the case when $\bar{s} = \infty$. From Corollary 2, optimal depletion policy is

$$q(t) = q(0)e^{-\frac{\delta}{1-\eta}t}\pi(Q(t))^{-\frac{1}{\eta}}.$$

Moreover, optimal q continuously declines (rises) forever accordingly as $h'(Q) > 0 (< 0)$. If $W(S)$ denotes the maximum expected life-time utility from optimally depleting the uncertain stock, it follows that

$$W(S) = q(0)^{\eta} \int_0^{\infty} e^{-\frac{\delta}{1-\eta}t}\, dt = q(0)^{\eta} \frac{1-\eta}{\delta}.$$

Next, if $\tilde{S}$ represents the certainty equivalent stock and, $V(\tilde{S})$, the corresponding maximum life-time utility from optimally depleting it over the same horizon as S, it is well known that optimal

$$\tilde{q}(t) = \tilde{q}(0)e^{-\delta(1-\eta)^{-1}t} \quad \text{and} \quad V(\tilde{S}) = \tilde{q}(0)^{\eta} \int_0^{\infty} e^{-\frac{\delta}{1-\eta}t}\, dt = \tilde{q}(0)^{\eta} \frac{1-\eta}{\delta}.$$

Since $W(S) = V(\tilde{S})$, by definition, we may infer that $q(0) = \tilde{q}(0)$ and $q(t) > \tilde{q}(t)$ for $t > 0$. Now invoking that $E(S) > \tilde{S}$ for all S would ensure the result, for $q(t)$ to optimally deplete $E(S)$ over the same time horizon is greater than that for $\tilde{S}$ for all t.

We turn next to the case when $\bar{s} < \infty$. In the light of the discussion in the text, we need examine only the situation of $h'(Q) > 0$ for all Q. Since Corollary 1 ensures that the optimal $q(t)$ declines continuously until exhaustion according to

$$q(t) = q(0)e^{-\frac{\delta}{1-\eta}t}\pi(Q(t))^{-\frac{1}{\eta}},$$

we may establish the result by proceeding in exactly the same manner as in the preceding case.

References

[1] P. Dasgupta, Some recent theoretical explorations in the economics of exhaustible resources, in: H. Gottinger, R. Radner, M. Yaari (Eds.), Systems Theory for Solving Environmental Problems, Academic Press, New York, NY, 1974.
[2] P. Dasgupta, G. Heal, The optimal depletion of exhaustible resources, Rev. Econ. Stud. Symp. (1974).
[3] R.J. Gilbert, Optimal depletion of an uncertain stock, Institute for Mathematical Studies in the Social Sciences, Discussion Paper No. 207, Stanford University, 1976.
[4] R.J. Gilbert, Optimal depletion of an uncertain stock, Rev. Econ. Stud. 46 (1979) 47–57.
[5] J. Highfill, M. McAsey, An optimal control problem in economics, Int. J. Math. Math. Sci. 14 (1991) 537–544.
[6] M. Hoel, Resource extraction when a future substitute has an uncertain cost, Rev. Econ. Stud. 45 (1978) 637–644.
[7] J.D. Kalbfleisch, R.L. Prentice, The Statistical Analysis of Failure Time Data, Wiley, New York, NY, 1980.
[8] M. Kemp, How to eat a cake of unknown size, in: M. Kemp (Ed.), Three Topics in the Theory of International Trade, North-Holland, Amsterdam, 1976.
[9] N.M. Kiefer, Economic duration data and hazard functions, J. Econ. Lit. 36 (1988) 646–679.
[10] J.P. Klein, M.L. Moeschberger, Survival Analysis: Techniques for Censored and Truncated Data, Springer, New York, 1997.
[11] T. Koopmans, Some observations on 'optimal' economic growth and exhaustible resources, in: H.C. Bos, H. Linnemann, P. de Wolff (Eds.), Economic Structure and Development, North-Holland, Amsterdam, 1973, pp. 239–255.

[12] T. Koopmans, Proof for a case where discounting advances the dooms day, Rev. Econ. Stud. Symp. (1974) 117–120.
[13] R.C. Kumar, How long to eat a cake of unknown size? Optimal time horizon under uncertainty, Can. J. Econ. 35 (2002) 843–853.
[14] R.C. Kumar, F.M. Naqib, On the determination of optimal time horizon in a control problem in natural resource economics, Int. J. Math. Math. Sci. 18 (1995) 617–620.
[15] T. Lancaster, The Economic Analysis of Transition Data, Cambridge University Press, Cambridge, 1990.
[16] E.T. Lee, Statistical Methods for Survival Data Analysis, Wiley, New York, 1992.
[17] G. Loury, The optimal exploitation of an unknown reserve, Rev. Econ. Stud. 45 (1978) 621–636.
[18] R.S. Pindyck, The optimal production of an exhaustible resource when price is exogenous and stochastic, Scand. J. Econ. 8 (1981) 277–288.
[19] N. Vousden, Basic theoretical issues of resource depletion, J. Econ. Theory 6 (1973) 123–143.