LN-12 Notes on the history of general equilibrium, welfare economics

*General equilibrium*

We have been through some parts of this already and know that the first comprehensive attempt of providing a theory to explain and determine prices for the whole economy was done by Léon Walras in *Elements of Pure Economics*. But the idea of general equilibrium may be traced further back and is certainly to be found in Adam Smith’s *WN*.

Walras approached this problem by moving from simpler cases to more comprehensive ones through a succession of models, each taking into account more aspects of a real economy, first two commodities, then many commodities, then adding production and finally also trying to dynamize the model to encompass growth and at last money.

Walras’ contribution is in a class by itself in the history of economics. He gave to economics the important paradigm of general equilibrium which turned out to be of enormous importance and by this he also left to successors a research program within which a number of the most brilliant economists in 20C have worked.

Walras left in fact a number of unsolved problems and inconsistencies, which is not strange in view of the time he presented his ideas. Walras intended to show the existence of an equilibrium, its uniqueness and its stability. But in this he failed, perhaps he thought he had solved it but he had not. Of course his general equilibrium paradigm had an immediate and ruling influence on thinking in economics.

On the existence problem Walras basically just counted equations. On stability he proposed a dynamic process by which general equilibrium might be reached through the so-called *tatonnement* process. This was a kind of hypothetical device, very ingeniously invented. In the process prices are announced, Walras said it was by an *auctioneer*, and agents stated how much of each good they would like to supply or demand at the prices announced by the auctioneer. No transactions and no production would by assumption take place at disequilibrium prices. Instead, prices are lowered for goods with positive prices and excess supply. Prices are raised for goods with excess demand.
Would the process terminate in equilibrium with demand equal to supply for goods with positive prices and demand not exceeding supply for goods with a price of zero? Walras was not able to provide a definitive answer to this question. A further question is: if the process converged would it to a uniquely determined equilibrium. The problems turned out to be a very difficult mathematical challenge to resolve.

Then what happened after Walras. Nothing much between Walras’ time and 1930. In the 1950s Arrow and Debreu dominated the scene by casting the problem in a framework of more abstract approach relying on more powerful mathematical tools. Another important name in this period is Lionel McKenzie. Many other also contributed in the 1950s and of course also later. We first look at what happened between 1930 and 1950.

Most of the development 1930-50 was done by Central European economists closely connected to a circuit of highly mathematical of oriented economists in Vienna in the 1920s.

The starting point was not Walras’ own formulation but that of Gustav Cassel who reformulated the equilibrium system in a simplified way, in which he postulated demand functions without reference to utility and with constant input coefficients in production. Let me briefly give you the Cassel system:

There are $n$ commodities and $m$ factors of production. Demand quantities are denoted $D_i$ ($i = 1,2,...,n$). Supply (production) quantities are $S_i$ ($i = 1,2,...,n$). Commodity prices are $p_i$ ($i = 1,2,...,n$). Resource prices are $q_j$ ($j = 1,2,...,m$). The given quantities of factors of production are $R_j$ ($j = 1,2,...,m$).

The equation system has four sets of equations:

1. $D_i = F_i(p_1,p_2,...,p_n)$ ($i = 1,2,...,n$) demand equations
2. $p_i = a_{i1}q_1 + a_{i2}q_2 + ... + a_{im}q_m$ ($i = 1,2,...,n$) price equations
3. $D_i = S_i$ ($i = 1,2,...,n$) demand equals supply
4. $R_j = a_{j1}S_1 + a_{j2}S_2 + ... + a_{jn}S_n$ ($j = 1,2,...,m$) resource demands

This system was in a widely read book by Cassel from around 1920. Notice that Cassel ignored the possibility of excess supply and zero price. He defended on
this point his system as being only concerned with scarce commodities. He also
had a “dynamized” version with all resources growing at the same rate.

In the 1920s Vienna was a center of intellectual and cultural activity attracting
many gifted persons from Central Europe and beyond. It was the venue of
logical positivism also known as the Wiener Kreis (Carnap, Neurath), other
philosophers (Popper, Wittgenstein), architecture, painting, and of course also
economics (Schumpeter, Morgenstern). Many highly gifted mathematician and
logicians (Tarski, Gödel) were there, also Ludwig von Mises’ brother Richard
von Mises and Carl Menger’s son Karl Menger.

It was in this fertile environment that the further development of general
equilibrium theory took off. Karl Menger had organized a Mathematical
Colloquium every fortnight and to this colloquium came – on and off –
Hungarian-born Karl Schlesinger (1889-1938) who was a banker with no
academic affiliation but also a very insightful economist, and Rumanian-born
Abraham Wald (1902-1950), other who frequented comprised Kurt Gödel,
sometimes also John von Neumann. Menger’s Colloquium can be viewed
drawing on three distinct intellectual strands: the logical positivism of the
Wiener Kreis, the mathematics of David Hilbert’s “formalist programme” and
the economics of the Austrian School. A key theme became the investigation of
the general equilibrium, perhaps more as metatheoretic system than purely for
its economics. Morgenstern who was director of a very well known business
cycle institute in Vienna and would team up with von Neumann in USA during
WWII may not have been good enough in mathematics to play a role in the
colloquium.

Schlesinger wrote an article, On the Production Equations of Economic Value
Theory (1933/1968), rewriting Cassel’s system introducing “complementary
slackness” conditions. Shortly afterwards Abraham Wald wrote two papers, On
the unique non-negative solvability of the new production equations
(1934/1968) and On the production equations of economic value theory
(1935/1968), which provided the first existence proof for a unique equilibrium
for the static Walras-Cassel system, using complementary slackness and
introducing the weak axiom of revealed preference. Wald wrote another
brilliant paper on this problem in Vienna but had then already been hired by
Morgenstern to work on statistical issues at the business cycle institute. Wald
continued to do brilliant work but had to flee Europe in 1938, joined the
Cowles Commission and opened a new field for himself in mathematical
statistics until his life was cut short by an airplane accident in 1950. Schlesinger
killed himself at Anschluss.
von Neumann worked in multiple fields and wrote an early paper titled *The Theory of Games* (1928/1959), a forerunner of the 1944 classic with Morgenstern. There are interesting connections between game theory and the general equilibrium problem (and also with linear programming which von Neumann also contributed towards). von Neumann moved to Princeton in 1931 but presented some years later to the Menger Colloquium a paper titled *On an economic equation system and a generalization of the Brouwer fixed point theorem* (1937/1945-46). The paper was on a multisectoral expanding economy, a problem somewhat similar to the general equilibrium problem. von Neumann was the first to use explicit fixed point techniques, explicit duality formulations, and convexity arguments, and characterized the as a saddle-point. The article has been called the single most important article in mathematical economics.

Menger’s Colloquium had thus been the venue for important breakthroughs in general equilibrium, but also for game theory, uncertainty and other topics. The Colloquium was disbanded after the German take-over in 1938 and most of the members were scattered in exile in Britain USA. Menger went to Notre Dame. Much of the colloquium’s research programme was taken up in the 1940s and 1950s by the Cowles Commission. It was from there and surroundings that the next steps were taken.

Gerard Debreu (1921-2004), French mathematician and economist, came with Rockefeller Fellowship to Cowles Commission in 1950 and remained at CC for ten years. Kenneth Arrow (1921- ) graduated in mathematics and got “hooked on” economics when he took Hotelling’s course in mathematical economics at Columbia. After four years war service Arrow came to Cowles Commission in April 1947 and moved to Stanford in 1949. At Cowles Commission Arrow originally intended to start on a PhD on general equilibrium but in some way he was diverted into the PhD dissertation wrote titled *Social Choice and Individual Values*, which became a classic in welfare analysis.

Arrow and Debreu thus were both at Cowles Commission but not at the same time. Debreu, naturally, had studied Walras in France, while Arrow’s general equilibrium background was primarily Hicks’ *Value and Capital*. In 1950 both Arrow and Debreu presented papers far away from each other, but foreshadowing their forthcoming contributions.

Debreu presented a paper titled *The Coefficient of Resource Utilization* (1951), which gave “a non-calculus proof of the intrinsic existence of price systems associated with the optimal complexes of physical resources – the basic
theorem of the new welfare economics. ... This proof is based on convexity properties” (Debreu 1951). Arrow presented a paper titled *An Extension of the Basic Theorem of Classical Welfare Economics* (1951) which proved that equilibrium outcomes are optimal using convex set theory, just as Debreu had done for existence.

The third name mentioned above was Lionel McKenzie (1919-2010) who graduated in economics who also had a year at Cowles Commission in 1949-50. McKenzie worked on a general model of international trade with $m$ countries, each capable of producing $n$ final goods, which led to a problem of general equilibrium for which McKenzie succeeded in proving existence by means of the Kakutani fixed-point theorem.

Neither Arrow, Debreu or McKenzie had by this stage properly digested or the results of Wald and von Neumann but so they seemed to have done quickly, and Arrow and Debreu entered at this stage into a cooperation which resulted in two papers, one by Debreu titled *A Social Equilibrium Existence Theorem* (1952) and a joint paper in *Econometrica* titled *Existence of an Equilibrium for a Competitive Economy* (1954). The 1954 paper is thus the original source for the Arrow-Debreu or Arrow-Debreu-McKenzie general equilibrium model and proves the existence of an equilibrium price vector for the economy described in fairly abstract mathematical terms.

Since then there has been much further development of general equilibrium ideas. The 1954 result still figures prominently as representing the modern Walrasian paradigm.

Arrow who has enormous achievements in many fields of economics was awarded the Nobel prize in economics jointly with John Hicks in 1972 for “their pioneering contributions to general economic equilibrium theory and welfare theory”, while Debreu who is less known for other things than his contribution to general equilibrium theory was awarded the Nobel Prize in 1983 for “having incorporated new analytical methods into economic theory and for his rigorous reformulation of general equilibrium.”

Then there is one more important but tricky result that need to be mentioned, not least as it can be read as a little disturbing. It is called the *Sonnenschein–Mantel–Debreu theorem*, named after Debreu, same as above, Rolf Ricardo Mantel (1934-1999), an Argentinian economist, and American economist Hugo Freund Sonnenschein (1941- ).
Let us first introduce $z_i(p)$ as the vector of excess demand for commodities for agent $i$ when the price vector is $p$. We then have $z(p) = \sum z_i(p)$ as the total excess demand for the economy.

For an equilibrium we need to find a price vector $p$ such that the elements of the excess demand vector $z(p)$ is zero or less than zero. The properties of $z(p)$ is in focus here. From the assumptions made about individual agents’ behaviour it follows that $z(p)$ will be continuous, homogeneous of degree zero, and obey Walras’ law.

These inherited properties are not sufficient to guarantee that the aggregate excess demand functions obey the weak axiom of revealed preference. These aggregate demand functions can have "any shape", which means that it is impossible to deduce from a maximizing behavior of households and firms the shape of their supplies and demands. This has serious consequences, the uniqueness and the stability of the equilibrium is not guaranteed. There may be more than one price vector at which excess demand is zero and no general process directing toward any equilibrium point can be deduced from the assumptions. The Sonnenschein–Mantel–Debreu theorem is referred to as the “Anything Goes Theorem”.

A reason for this awkward results can in economic terms be said to be the presence of wealth effects. A change in a price of some particular good has two consequences, the good in question becomes cheaper or more expensive relative to all other goods, with increase/decrease in the demand, at the same time the price change also affects the real wealth of consumers, making some richer some poorer, resulting in increased/reduced demand. Substitution effects and wealth effects can work in opposite or reinforcing directions, which means that more than one set of prices can clear all markets simultaneously.

It is still true as it was for Walras that the number of equations is equal to the number of prices but there is no guarantee of a unique solution. There are several things to be noted. First, even though there may be multiple equilibria, every equilibrium is still guaranteed, under standard assumptions, to be Pareto efficient. However, the different equilibria are likely to have different distributional implications and may be ranked differently by any given social welfare functions. It can also be shown (under some conditions) that the number of equilibria will be finite and all of them will be locally unique. This means that comparative statics can still be relevant as long as the shocks are not too large. But this leaves the question of the stability of the equilibrium
unanswered as a comparative statics point of view does not allow to know what happen when one moves from one equilibrium.

Some critics have taken the theorem to mean that General equilibrium analysis cannot be usefully applied to understand real life economies since it makes imprecise predictions (i.e. “Anything Goes”), and as a “deeply negative result” for microeconomic research, as microeconomic rationality assumptions have no equivalent macroeconomic implications. Others say that there is no a priori reason why one should expect a real life economy to have a unique equilibrium and hence the possibility of multiple outcomes is in fact a realistic feature of the theory.

**Welfare Economics**

What is ‘welfare economics’? It can be said to having to do with defining and measuring the ‘welfare of society’. It can also be said to be concerned with which economic policies lead to optimal outcomes. It is any case related to general equilibrium and the properties of a market system. Historically it is rooted in Adam Smith’s discussion of the consequences of market rule but the problems have addressed within welfare economics have changed over time. Thus it is a kind of applied theoretic branch.

The most common conception of Adam Smith’s view on market and welfare can be summarized as: (a) the principal human motive is self-interest; (b) the invisible hand of competition automatically transforms the self-interest of many into the common good; (c) therefore, the best government policy for the growth of a nation’s wealth is that policy which governs least.

We know that Smith's arguments were at the time directed against the mercantilists but they have been used and reused by proponents of laissez-faire over the following 200+ years and very much present today in large paretsof the world as implicit in economic policies involving deregulation, tax reduction, denationalizing industries, and containment of government growth, and indeed in the deliberate restoration of private markets in former centrally planned countries.

Throughout the time since Smith there have also been proponents of opposing views, intellectuals, philosophers and indeed also economists who believe that: (a) economic planning is superior to laissez-faire; (b) markets are often monopolized in the absence of government intervention, crippling the invisible
hand of competition; (c) even if markets are competitive, the existence of external effects, public goods, information asymmetries and other market failures ensure that laissez-faire will not bring about the common good; (d) and in any case, laissez-faire may produce an intolerable degree of inequality.

The subfield of economics called welfare economics has dealt with issues of this fundamental debate that can be traced back to Adam Smith. Three key questions running through the history of welfare economics are the following: Will an equilibrium outcome in a competitive economy be optimal, as suggested by Smith? Can any optimal outcome be achieved by a modified market mechanism? Is there a reliable way to measure social welfare, or to derive the preferences of society from the preferences of individuals? We recognize the first two questions as closely related to the first and second welfare theorem.

In *WN* Adam Smith conveyed in a persuasive way the impression that laissez-faire leads to the common good and the “invisible hand” metaphor is (too?) often quoted. In fact what Smith wrote seemed primarily to suggest that laissez-faire would render the annual revenue of the society as great as possible, to maximize it in other words.

The first fundamental theorem of welfare economics an optimality character of the outcome can thus be traced back to Smith. But what is the ‘common good’? Following Smith we would look for a measure of total value of goods and services produced in the economy. Marshall’s closest student and follower, Arthur Pigou, wrote in 1920, following Smith, that the “free play of self-interest” leads to the greatest “national dividend”.

We have followed the idea further on via Walras, Edgeworth, Pareto to Arrow (1951), increasingly cast into the more modern mathematical language of economics was invented. Also a couple of predecessors of Arrow (1951) ought to be mentioned: Abba Lerner (1934) and Oscar Lange (1942). They had simpler models within which they proved versions of the first fundamental theorem of welfare economics.

The modern interpretation of ‘common good’ is Pareto optimality, rather than maximized gross national product:

*First fundamental theorem of welfare economics:* Assume that all individuals and firms are self-interested price takers. Then a competitive
equilibrium is Pareto optimal.

When ultimate consumers appear in the model Pareto optimality means that there is no feasible alternative that makes everyone better off. Pareto optimality is thus a dominance concept based on comparisons of vectors of utilities. It rejects the notion that utilities of different individuals can be compared, or that utilities of different individuals can be summed up and two alternative situations compared by looking at summed utilities.

Then concept of Pareto optimality can be used also when ultimate consumers do not appear in the model, a situation is said to be Pareto optimal if there is no alternative that results in the production of more of some output, or the use of less of some input, all else equal. Obviously, saying that a situation is Pareto optimal is not the same as saying it maximizes GNP, or that it is best in some unique sense. There are generally many Pareto optima.

Does that make Pareto optimality less relevant? In spite of the multiplicity of optima in a general equilibrium model, most states are non-optimal. If a state is chosen at random the chance of hitting an optimum would be zero. Therefore, the theorem says a lot. A Pareto optimality is a common good concept in the sense that it can get common assent, no one would argue that society should settle for a situation that is not optimal, because then there exists a better state that all prefer.

The first fundamental theorem of welfare economics is mathematically true but nevertheless open to objections of different kinds, such as the following:

First, the theorem is an abstraction that ignores the facts. Preferences of consumers are not given, they are created by advertising. The real economy is never in equilibrium, most markets are characterized by excess supply or excess demand, and are in a constant state of flux. The economy is dynamic, tastes and technology are constantly changing, whereas the model assumes they are fixed. The cast of characters in the real economy is constantly changing, the model assumes it fixed.

Second, the theorem assumes competitive behaviour, whereas the real world is full of monopoly and market power.

Third, the theorem assumes there are no externalities. In fact, if in an exchange economy person 1's utility depends on person 2's consumption as well as his
own, the theorem does not hold. Similarly, if in a production economy one firm's production possibility set depends on the production vector of some other firm, the theorem breaks down. The theorem further assumes away public goods, like national defence, judicial systems or lighthouses, that are necessarily non-exclusive in use. If such goods are privately provided (as they would have to be in a completely laissez-faire economy), their level of production will be suboptimal.

Fourth, and perhaps most important, the theorem ignores distribution. Laissez-faire may produce a Pareto optimal outcome, but there are many different Pareto optima, and some are fairer than others. Some people are endowed with resources that make them extremely rich, while others, through no fault of their own, are extremely poor.

We leave the first and second objections aside. The third, on externalities and public goods, are well acknowledged and dealt with by appropriate modifications of the market mechanism, such as Pigouvian taxes on harmful externalities, or appropriate Coasian legal entitlements (property rights) to resources such as rivers, clean air.

Pigou's remedy was for the state to eliminate the divergence between trade and social net product by imposing appropriate taxes (or, in the case of beneficial externalities, bounties). The Pigouvian tax would be set equal to marginal external cost, Optimality would be re-established. Coase's contribution was to emphasize the reciprocal nature of externalities and to suggest remedies based on common law doctrines. In his view the polluter damages the pollutee only because of their proximity; for example, the smoking factory harms the other only if it happens to locate close downwind. Coase rejects the notion that the state must step in and tax the polluter.

With respect to public goods, Samuelson (1954) derived formal optimality conditions for their provision, the issue has received much attention from economists. An earlier Wicksellian idea, conveyed through Lindahl (1919), where an individual's tax is set equal to his marginal benefit, is another approach.

But what about the fourth objection - distribution?

The historical literature diverges on the question of rectifying the distributional inequities of laissez-faire. One way is the command economy approach:
central bureaucracy makes detailed decisions about consumption decisions and production decisions, as a consequence the command approach fails to create appropriate incentives for individuals and firms.

The other way is to solve distribution problems by transferring income or purchasing power among individuals, and then let the market work. The only kind of purchasing power transfer not causing incentive-related losses is, however, the lump-sum money transfer.

Then we have come close to the second theorem of welfare economics establishing that the market mechanism, modified by the addition of lump-sum transfers, can achieve virtually any desired optimal distribution. In formal terms and under somewhat more stringent conditions than necessary for the first theorem, including assumptions regarding quasi-concavity of utility functions and convexity of production possibility sets, the second theorem is:

Second fundamental theorem of welfare economics: Assume that all individuals and producers are self-interested price takers. Then almost any Pareto optimal equilibrium can be achieved via the competitive mechanism, provided appropriate lump-sum taxes and transfers are imposed on individuals and firms.

The second theorem is relevant for an old debate about the feasibility of socialism. Anti-socialists including von Mises (1922) had argued that informational problems would make it impossible to coordinate production in a socialist economy. This was discussed in the 1930s and contradicted by a number of pro-socialist contributor, particularly Oscar Lange, argued that those problems could be overcome by a central planning board, which limited its role to merely announcing a price vector. This was called ‘decentralized socialism’. Given the prices, managers of production units would act like their capitalist counterparts; in essence, they would maximize profits. By choosing the price vectors appropriately, the central planning board could achieve any optimal production plan it wished.

In real societies there is never been a straight choice between a laissez-faire economy and a command economy. The choices are more modest. When choosing among alternative tax policies, or trade and tariff policies, or development policies, or anti-monopoly policies, or labour policies, or transfer policies, what shall guide the choice? The applied welfare economist's advice is usually based on some notion of increasing total output in the economy. The
practical political decision, in a democracy, is normally based on voting.

The applied welfare economist usually focuses on ways to increase total output, ‘the size of the pie’, or at least to measure changes in the size of the pie, or in our economist Prime Minister’s social democratic lingo it is “Skape og dele”.

This sounds simple enough but theory suggests that the pie cannot be easily measured - for a number of reasons. To start, any measure of total output is a scalar, a single number. If the number is found by adding up utility levels for different individuals, illegitimate interpersonal utility comparisons are being made. If the number is found by adding up the values of aggregate net outputs of all goods, there is an index number problem. The value of a production plan will depend on the price vector at which it is evaluated. But, in a general equilibrium context, the price vector will depend on the aggregate net output vector, which will in turn depend on the distribution of ownership or wealth among individuals.

An early and crucial contribution to the analysis of whether or not the economic pie has increased in size was made by Nicholas Kaldor, a Hungarian-born British economist, in 1939. Kaldor argued, in fact with reference to the repeal of the Corn Laws in England, a non-Pareto change can be justified on the grounds that the winners might (in theory) compensate the losers: “… it is quite sufficient [for the economist] to show that even if all those who suffer as a result are fully compensated for their loss, the rest of the community will still be better off than before”. But Tibor Scitovsky, a Hungarian-born American economist pointed out already in 1941 that Kaldor's compensation criterion (as well as related criterion proposed by Hicks) was inconsistent.

Consider a move from situation A to situation B. It is possible to judge B Kaldor superior to A (the move is an improvement) and simultaneously judge A Kaldor superior to B (the move back would also be an improvement). This Scitovsky paradox can be avoided via a two-edged compensation test, according to which B is judged better than A if (a) the potential gainers in the move from A to B could compensate the potential losers, and still remain better off, and (b) the potential losers could not bribe the gainers to forgo the move. However, while Scitovsky's two-edged criterion has some logical appeal, it still has a major drawback: it ignores distribution. Therefore, it can make no judgement about alternative distributions of the same size pie. Even worse, both the Kaldor and the Scitovsky criteria would approve of a change that makes the wealthiest
man in society richer by $1 billion, while making each of the million poorest people worse off by $999.

Another important tool for measuring changes in economic welfare is the concept of consumer's surplus, as introduced by Marshall, defined as the difference between what an individual would be willing to pay for an object, at most, and what he actually does pay. With a little faith, the economic analyst can measure aggregate consumers’ surplus as the area under a demand curve, and this is indeed used in order to evaluate changes in economic policy. The applied welfare economist attempts to judge whether the pie would grow in a move from A to B by examining the change in consumers’ surplus (plus profits, if they enter the analysis). Some faith is required because consumers’ surplus, like the Kaldor criterion, is theoretically inconsistent for reasons we cannot pursue here (see Boadway 1974). Under certain specific assumptions about the individual utility functions the consumers’ surplus can be relied upon.

As noted above, although the tools of applied welfare economics are widely used and very important in practice, in theory they should be viewed with some skepticism.

The second theorem itself raises questions about distribution that many would view as essentially political. How should the distribution of income be chosen? How can the best distribution of income be chosen from among many Pareto optimal ones? Majority rule is a commonly used method of choice in a democracy.

The central theoretical problem with majority voting has been known since the time of Condorcet’s *Essai sur l’application de l’analyse à la probabilité des décisions rendues à la pluralité des voix* (1785), namely that voting may be logically inconsistent. The standard Condorcet voting paradox assumes three individuals 1, 2 and 3, and three alternatives x, y and z, where the three voters have the following preferences, as set out in Sandmo:

1: x y z
2: y z x
3: z x y

Such voting cycles are not peculiar. This can be illustrated by taking the alternatives to be different distributions of one economic pie. Suppose, in other words, that the distributional issues are to be solved by majority voting, and
assume that what is to be divided is a fixed total of wealth, say 100 units. Now let \( x \) be 50 units for person 1, 30 units for person 2 and 20 units for person 3, i.e. \( x = (50,20,30) \). Similarly, let \( y = (30,50,20) \) and \( z = (20,30,50) \). The result is that our three individuals have precisely the voting paradox preferences. It turns out that all the distributions of 100 units of wealth are connected by endless voting cycles. The reality of voting cycles ia an awkward obstacle for to the economist inclined to recommend legislation about economic choices, especially among alternative distributions of income or wealth.

How then can economic choices be made and distribution problems be solved? One potential answer is to assert the existence of an economic welfare function as introduced by Abram Bergson (1914-2003), particularly as the Bergson-Samuelson welfare function where the arguments are individual utility. Bergson declared the object as "to state in precise form the value judgments required for the derivation of the conditions of maximum economic welfare".

Another approach was launched by Kenneth Arrow in his dissertation *Social Choice and Individual Values* (1951). Arrow was very familiar with all the ideas mentioned above. Arrow's analysis led up to a theorem which can be viewed in several ways: as a statement about the distributional questions raised by the first and second theorems; as an extension of the Condorcet voting paradox; as a statement about the logic of voting; and as a statement about the logic of Bergson welfare functions, compensation tests, consumers' surplus tests, and indeed all the tools of the applied welfare economist. Arrow's theorem has by some been called, and for justifiable reasons, the third fundamental theorem of welfare economics.

Arrow's analysis is at a high level of abstraction. It is assumed a given set of at least three distinct alternatives, which might be allocations in an exchange economy, distributions of wealth, etc. The alternatives are called \( x, y, z \) and so on. Also assumed is a fixed society of individuals, numbered 1, 2,..., \( n \). Each individual has preferences, \( R_i \), over the alternatives. A *preference profile* for society is a specification of preferences for each individual, \( R = (R_1,R_2,...,R_n) \) and \( R_s \) is the society's preference relation, arrived at in a way yet to be specified. Arrow was concerned with the logic of how individual preferences are transformed into social preferences. How can \( R_s \) be derived from the individual preferences.

Arrow stated certain rationality or reasonability conditions that the individual
preference relations had to fulfill. We can look at these conditions in Amartya Sen’s version (1970).

1. Universality. The function should always work, no matter what individual preferences might be. It would not be satisfactory, for example, to require unanimous agreement among all the individuals before determining social preferences.

2. Pareto consistency. Social preferences should be consistent with the Pareto criterion. That is, if everyone prefers $x$ to $y$, then the social preference is $x$ over $y$.

3. Independence. Suppose there are two alternative preference profiles for individuals in society, but suppose individual preferences regarding $x$ and $y$ are exactly the same under the two alternatives. Then the social preference regarding $x$ and $y$ must be exactly the same under the two alternatives. In particular, if individuals change their minds about a third ‘irrelevant’ alternative, this should not affect the social preference regarding $x$ and $y$.

4. Non-dictatorship. There should not be a dictator. In Arrow's abstract model, person $i$ is a dictator if society always prefers what he prefers; that is, if $i$ prefers $x$ to $y$, then the social preference is $x$ over $y$.

An economist or policymaker who wants an ultimate answer to questions involving distributions, or involving choices among alternatives that are not comparable under the Pareto criterion, could use an Arrow SWF for guidance. Unfortunately, Arrow showed that imposing conditions 1 to 4 guarantees that Arrow functions do not exist:

Third fundamental theorem of welfare economics: There is no Arrow social welfare function that satisfies the conditions of universality, Pareto consistency, independence, non-dictatorship.

This theorem tells us that we cannot find an Arrow social welfare function satisfying certain reasonable requirements. An Arrow function maps preference profiles (that is, preference relations for each and every member of society) into social preference relations. But in order to make social choices, it is not really necessary to have a social preference relation. We could have just a rule that tells us, if the set of alternatives is $x$, $y$, $z$ and so on, and the preference profile is $R = (R_1,R_2,...,R_n)$, then the best alternative is such-and-such, a mapping from preference profiles into alternatives, written symbolically as:
Such a rule is called a *social choice function* (SCF). An Arrow function produces a social ranking of all alternatives; an SCF just produces a winner. As an example, think of plurality voting, with some kind of rule to break ties.

The difficulty with SCFs is that they may create obvious incentives for people to misrepresent their preferences, so as to obtain better (for them) social choices. Consider again the Condorcet voting paradox preferences:

1: x y z
2: y z x
3: z x y

Suppose plurality voting and, in case of a tie, the social choice is the outcome closest to the beginning of the alphabet. Under this rule, if 1 votes for his favourite, x, and persons 2 and 3 do likewise, there is a three-way tie, which is resolved with the (alphabetical) choice of x. Now consider 2’s situation and you can easily figure out, if persons 1 and 3 continue to vote for their favourites, person 2 can switch from his favourite y to his second favourite z. Then social choice changes, from x to z, a better outcome for 2.

Reporting a false preference relation in order to bring about an SCF outcome that you prefer to the one you get if you are honest, is called *strategic behaviour*. It is obviously a bad thing if an SCF produces lots of opportunities for strategic behaviour, then there is no reason to believe that the outcome, based as it is on false reports, is truly best for society. If an SCF has the property that it is never advantageous for anyone to report a false preference relation it is called *strategy-proof*. There has been some further development along this line of reasoning, that is about SCFs that are immune to strategic behaviour, and that satisfy a few other reasonable conditions.

We leave the issue here, only mention that there is a theorem called the Gibbard–Satterthwaite theorem from around 1975 and is logically very close to third fundamental theorem of welfare economics. It can be stated as:

*Gibbard–Satterthwaite theorem:* There is no social choice function that satisfies the conditions of universality, non-degeneracy, strategy-proofness and non-dictatorship.
Then what is the conclusion if any? The first and second theorems were encouraging results suggesting that the market mechanism has great virtue. The third theorem exposes impossibilities and paradoxes in economic choices, voting choices, and, in general, almost any choices made collectively by society. The Gibbard–Satterthwaite theorem is a further negative result: any plausible social choice function will, under some circumstances, produce incentives for someone to lie. More recent work by Erik Maskin provides perhaps some hope. His results suggest a way for a social planner to design a game, whose Nash equilibria will implement a desired social choice function.

Maskin got the Nobel Prize in 2007 (shared with Leo Hurwicz and Roger Myerson) for “for having laid the foundations of mechanism design theory”.