HG
24.01.2014

Answers for the first seminar

Exercise 1

1A. Clearly, $E(Y_t)=0$, and, for $h>1$, $\text{cov}(Y_t,Y_{t+h})=\text{cov}(Y_t,Y_{t-h})=E(Y_tY_{t-h})=0$

$$\gamma(0)=\text{var}(\varepsilon_t+\theta\varepsilon_{t-1})=(1+\theta^2)\sigma_\varepsilon^2$$

$$\gamma(1)=E(\varepsilon_t+\theta\varepsilon_{t-1})(\varepsilon_{t-1}+\theta\varepsilon_{t-2})=\text{E}(\theta\varepsilon_{t-1}^2)=\theta\sigma_\varepsilon^2$$

1B. From **A** we get,

$$\gamma_\eta(0)=(1+\delta^2)\sigma_\eta^2=\left(1+\frac{1}{\theta^2}\right)\theta^2\sigma_\varepsilon^2=\gamma(0)$$

$$\gamma_\eta(1)=\delta\sigma_\eta^2=\frac{1}{\theta}\theta^2\sigma_\varepsilon^2=\theta\sigma_\varepsilon^2=\gamma(1)$$

1C. $\{\varepsilon_t\}$ are *iid* and normally distributed, implies that the joint distribution of any set of $Y_t,Y_{t+1},\dots,\text{Y}_{t+h}$ is determined by $\{\gamma(0),\gamma(1)\}$.

1D. The companion power series of $\psi(L)=\dfrac{1}{1+\theta L}$ is

$\psi(z)=\dfrac{1}{1+\theta z}=\dfrac{1}{1-(-\theta)z}=1+(-\theta)z+(-\theta)^2+\cdots$ (a geometric series) is well defined if and only if $|\theta z|=|\theta||z|<1$, i.e., $\psi(z)$ is well defined as a power series in *z* for all *z* such that $|z|\le 1$ if and only if $|\theta|<1$. This is then the criterion for invertibility in the example.

Exercise 2.

2A. $\dfrac{1}{1+2i}=\dfrac{1-2i}{(1+2i)(1-2i)}=\dfrac{1-2i}{1+2^2}=\dfrac{1}{5}(1-2i)$. Similarly for the other one.

2B. $|1\pm 2i|=\sqrt{1+4}=\sqrt{5}>1$, implying that the polynomial, $\varphi(z)=z^2-2z+5$, must be the companion polynomial for a causal AR(2) process. Hence, since $\varphi_0=2$, the difference equation specification for $\{Y_t\}$ must be

$$5Y_t-2Y_{t-1}+Y_{t-2}=\text{constant + WN-error}$$

Dividing the equation by 5, we get the AR(2) specification

(1) $Y_t - 0.4Y_{t-1} + 0.2Y_{t-2} = 2 + \varepsilon_t$ where $\varepsilon_t \sim WN(0,\sigma^2)$

Hence, we have $\varphi_0 = 2,\quad \varphi_1 = 0.4,$ and $\varphi_2 = -0.2$. Since dividing by a constant does not change the roots of a polynomial, the roots of the reversed companion polynomial are

(2) $r_1 = \frac{1}{5} + \frac{2}{5}i$ and $r_2 = \frac{1}{5} - \frac{2}{5}i$

which are both inside the unit circle.

The causal filter solution is now

$$\psi_t = c_1 r_1^t + c_2 r_2^t, \quad \text{where } c_1 = \frac{\varphi_1 - r_2}{r_1 - r_2}, \quad c_2 = \frac{r_1 - \varphi_1}{r_1 - r_2}$$

We have,

$$c_1 = \frac{0.4 - \frac{1}{5} + \frac{2}{5}i}{\frac{4}{5}i} = \frac{\left(\frac{1}{5} + \frac{2}{5}i\right)i}{-\frac{4}{5}} = \frac{i-2}{-4} = \frac{1}{4}(2-i)$$

Similarly

$c_2 = \frac{1}{4}(2+i)$ showing that $c_2 = \overline{c_1}$. Since also $r_2 = \overline{r_1}$, we must have

(3) $\psi_t = c_1 r_1^t + \overline{c_1 r_1^t}$ for $t = 0,1,2,\dots$

showing that ψ_t must be real (which is always the case). (If $d = a + ib$ is any complex number, $d + \overline{d} = 2a$ must be real.)

The long run effect is

$\psi(1) = \frac{1}{1-\varphi_1-\varphi_2} = \frac{1}{1-0.4+0.2} = 1.25$, and the expectation

$E(Y_t) = \frac{\varphi_0}{1-\varphi_1-\varphi_2} = 2\frac{5}{4} = 2.5$

2.C.

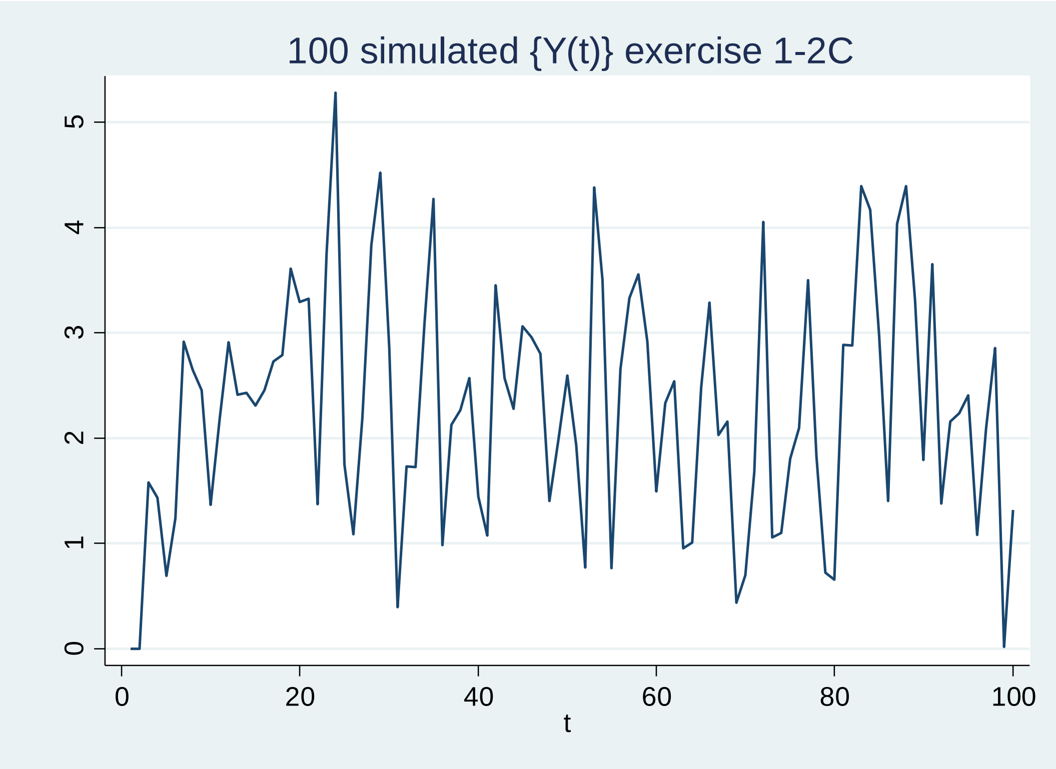
We must write the equation for Y_t as

$Y_t = 2 + 0.4Y_{t-1} - 0.2Y_{t-2} + \varepsilon_t$

to get the coefficients for Stata.

```
. set obs  100
obs was 0, now 100

. gen y=0

. gen t=_n

. tsset
time variable not set, use -tsset varname ..
> .-
r(111);

. tsset t
        time variable:  t, 1 to 100
                delta:  1 unit

. gen eps=rnormal()

. matrix y=(1,0.4,-.2,1)

. matrix coleq y=y:_cons y:L.y y:L2.y y:eps

. matrix list y

y[1,4]
        y:     y:     y:     y:
               L.    L2.
    _cons      y      y    eps
r1      1     .4    -.2      1

. forecast create
  Forecast model started.

. forecast coefvector y
  Forecast model now contains 1 endogenous
  variable.

. forecast solve, begin(3) log(off)

Computing dynamic forecasts for current mode
> l.
--------------------------------------------
> --
Starting period:  3
Ending period:    100
Forecast prefix:  f_

Forecast 1 variable spanning 98 periods.
----------------------------------------

list y f_y if t<=10

     +--------------+
     | y        f_y |
     |--------------|
  1. | 0          0 |
  2. | 0          0 |           (It would have been better, maybe, to set
  3. | 0   1.578344 |           the two first values of y = 2.5 = E(Y)     )
  4. | 0   1.431136 |
  5. | 0   .6907607 |
     |--------------|
  6. | 0   1.240486 |
  7. | 0    2.91584 |
  8. | 0   2.651467 |
  9. | 0   2.457587 |
 10. | 0   1.369082 |
     +--------------+
```

```
. tsline f_y
```

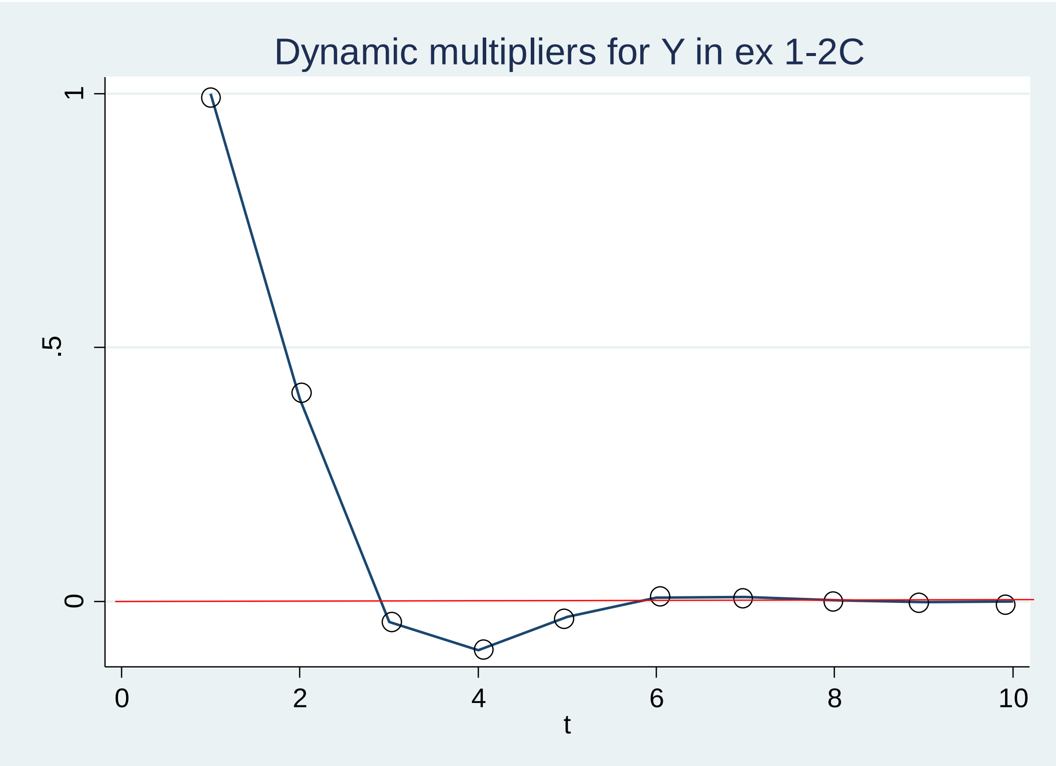


There seem to be some large low-frequency waves present (cycles?). Notice that it is not stationary from the start ($Y_1 = Y_2 = 0$).

The dynamic multipliers ψ_t :

```
gen psi=0

. replace psi = 1 in 1
(1 real change made)

. replace psi = .4 in 2
(1 real change made)

. matrix psi=(.4,-.2)

. forecast create,replace
  (Forecast model ended.)
  Forecast model started.

. matrix coleq psi= psi:L.psi psi:L2.psi

. matrix list psi

psi[1,2]
     psi:  psi:
       L.   L2.
     psi   psi
r1    .4   -.2

. forecast coefvector psi
  Forecast model now contains 1 endogenous
  variable.
```

```
. forecast solve, begin(3) log(off)

Computing dynamic forecasts for current mode
> l.
--------------------------------------------
> --
Starting period:  3
Ending period:    100
Forecast prefix:  f_

Forecast 1 variable spanning 98 periods.
----------------------------------------


. list f_psi if t<=10

     +-----------+
     |     f_psi |
     |-----------|
  1. |         1 |
  2. |        .4 |
  3. |      -.04 |
  4. |     -.096 |
  5. |    -.0304 |
     |-----------|
  6. |    .00704 |
  7. |   .008896 |
  8. |  .0021504 |
  9. |  -.000919 |
 10. | -.0007977 |
     +-----------+
```



Exercise 3.

A. We have

$$\varphi(L)y_t = \varphi(L)(Y_t-\mu) = \varphi(L)Y_t - \varphi(1)\mu = \varphi_0 + \frac{\theta(L)}{\varphi(L)}\varepsilon_t - \varphi(1)\frac{\varphi_0}{\varphi(1)} = \frac{\theta(L)}{\varphi(L)}\varepsilon_t$$

B. The only rule I use in the argument is that products of lag polynomials are commutative, i.e., so that

$$(1-L)(1-0.5L)(1+0.8L) = (1-0.5L)(1+0.8L)(1-L) = (1-0.5L)(1+0.8L)\Delta$$

Note. I should perhaps have added the note in **B** that a process like $Y_t = \Delta Y_0 + u_1 + u_2 + \cdots + u_t$, where $\{u_t\} = \{\Delta Y_t\}$ is stationary (and not itself the difference of a stationary process), is a called *an integrated process of order* 1 (written $Y \sim I(1)$), a concept basic in co-integration analysis. It can also be described as an ARIMA(p,1,q), in the case $u_t = \Delta\,\mathrm{Y}_t$ is an $\mathrm{ARMA}(p,q)$ process. In general, the notation, $Y_t \sim \mathrm{ARIMA}(p,d,q)$ means simply that $\Delta^d Y_t \sim \mathrm{ARMA}(p,q)$.

C.

(a) $Y_t = 1.5Y_{t-1} - 0.5Y_{t-2} + \varepsilon_t$

The companion polynomial, $1-1.5z+0.5z^2$ has the roots, 2 and 1 (implying roots 0.5 and 1 for the reversed comp. pol., and $\varphi(L) = (1-0.5L)(1-L)$. Hence Y_t is non-stationary, ΔY_t is causal AR(1). One root on the unit circle. (Y_t is a random walk, integrated of order 1, $Y_t \sim I(1)$).

(b) $Y_t = Y_{t-2} + \varepsilon_t$

The companion polynomial, $1-z^2 = (1-z)(1+z)$, has both roots on the unit circle, and $\varphi(L) = (1-L)(1-(-1)\mathrm{L})$. It is non-stationary and has to be differenced twice to be brought to stationarity, i.e., it is I(2).

(**Note.** It is,actually, a little more complicated than simple white noise integrated twice. If you take out every second Y_t from the sequence, you get an I(1) random walk, but if you include the whole time series, you get an I(2) (see appendix for more details if you are interested).

Another way, perhaps more interesting, to look at this is noting that $\{Y_t - Y_{t-2}\}$ is stationary. Defining $X_t = Y_{t-2}$, the two (non-stationary) time series, $\{Y_t\}$ and $\{X_t\}$ are (so called) *co-integrated,* which means that there is a linear combination of them (here $Y_t - X_t$) that is stationary (anticipating Ragnar's treatment later)). This is in line with the common use, interpreting (b) as a *seasonal* model, e.g., if the time unit is half-years, then (b) implicitly states that the first half of the year may give different type of response (Y_t) than the second half. If, for another example, the time unit is quarter, then the lag polynomial $(1-0.6L)(1-0.8L^4)$ would represent a causal stationary *seasonal* model, and the model $(1-0.6L)(1-L^4)\,\mathrm{Y}_t = \varepsilon_t$ would say that the difference $Y_t - Y_{t-4}$ is an AR(1) process.)

(c) $Y_t = 2Y_{t-1} - Y_{t-2} + \varepsilon_t$

The companion polynomial, $1-2z+z^2 = (1-z)^2$, i.e., two equal roots on the unit circle. Non stationary with $\Delta^2 Y_t = \varepsilon_t$. Can also be described as ARIMA(0,2,0).

(d) $Y_t = Y_{t-1} + 0.25Y_{t-2} - 0.25Y_{t-3} + \varepsilon_t$

The companion polynomial is $1 - z - 0.25z^2 + 0.25z^3$, which, using, e.g., the `polyroots` command in Stata, has the roots, -2, 2, 1. Hence, again non-stationary, with one root on the unit circle.

On the other hand, you may have noticed that the difference equation can be written, $\Delta Y_t = 0.25\Delta Y_{t-1} + \varepsilon_t$, i.e., $\{\Delta Y_t\}$ is causal and AR(1), or, in other words, $Y_t \sim \text{ARIMA}(1,1,0)$.

So all the examples in (a)-(d) are non-stationary with at least one root on the unit circle.

Appendix to 3C(b) (for those interested)

Suppose, e.g., $t = 2p$, an even number. By successive substitution we get

$$
\begin{aligned}
Y_{2p} &= \varepsilon_{2p} + \varepsilon_{2(p-1)} + \varepsilon_{2(p-2)} + \cdots + \varepsilon_2 + Y_0 \\
Y_{2p+1} &= \varepsilon_{2p+1} + \varepsilon_{2(p-1)+1} + \varepsilon_{2(p-2)+1} + \cdots + \varepsilon_1 + Y_{-1}
\end{aligned}
$$

Taking the difference, we get

$$
\begin{aligned}
&\Delta Y_{2p+1} = (\varepsilon_{2p+1} - \varepsilon_{2p}) + (\varepsilon_{2p-1} - \varepsilon_{2p-2}) + \cdots + (\varepsilon_3 - \varepsilon_2) + \varepsilon_1 + \Delta Y_0 = \\
&\eta_p + \eta_{p-1} + \eta_{p-2} + \cdots + \eta_1 + \varepsilon_1 + \Delta Y_0
\end{aligned}
$$

where $\eta_t = \varepsilon_{2t+1} - \varepsilon_{2t}$, implying that $\eta_1, \eta_2, \ldots$, are uncorrelated, with expectation 0 and constant variance, and, therefore white noise. Hence $\{\Delta Y_t\}$ is a random walk the can be reduced to stationarity by differencing once more, i.e., $\Delta_2 \Delta Y_t \sim$ stationary, where Δ_2 means $1-L^2$. In practice, however, we only need to difference once, since, $\Delta_2 Y_t$ is stationary! Note that this discussion on ΔY_t is rather academic since the seasonal interpretation mentioned in the answer has more practical relevance.