HG
Jan 2014 ECON 5101

Exercises I - 27 Jan 2014

Exercise 1.

A. Suppose $\{Y_t\}$ is a moving average process of order 1 (MA(1)) with expectation 0.

(1) $Y_t = \varepsilon_t + \theta\varepsilon_{t-1}$ where $\varepsilon_t \sim WN(0, \sigma_\varepsilon^2)$ (white noise with expectation 0 and variance, σ_ε^2) and $\theta \neq 1$. Show that the covariance function, $\gamma(h), \quad h = 0, \pm 1, \pm 2, \ldots$, is given by

$$\gamma(-h) = \gamma(h) = \begin{cases} (1+\theta^2)\sigma_\varepsilon^2 & h = 0 \\ \theta\sigma_\varepsilon^2 & h = 1 \\ 0 & h > 1 \end{cases},$$

and explain why $\{Y_t\}$ is stationary. Explain also why the autocorrelation function (acf) is

$$\rho(-h) = \rho(h) = \begin{cases} 1 & h = 0 \\ \dfrac{\theta}{(1+\theta^2)} & h = 1 \\ 0 & h > 1 \end{cases},$$

B. **The MA-representation in (1) is not unique!** Let $\eta_t = \theta\varepsilon_t$. Then also $\{\eta_t\}$ is white noise, $\eta_t \sim WN(0,\ \sigma_\eta^2 = \theta^2\sigma_\varepsilon^2)$. Let $\delta = 1/\theta$.
Show that the MA(1) process

$$Z_t = \eta_1 + \delta\eta_{t-1}$$

has the same autocovariance function as $\{Y_t\}$.

C. We say that some parameters in an econometric model are *not identified* if there are several values of the parameters that determine *the same* joint distribution of the observable variables. Explain why, assuming that $\{\varepsilon_t\}$ are *iid* and normally distributed, $\varepsilon_t \sim N(0,\sigma_\varepsilon^2)$, the results in **A** and **B** show that the parameters, $(\theta,\sigma_\varepsilon^2)$ in the MA-model (1) are not identified.

D. **Introduction.** This is a general property of MA-time series, i.e., that there are (usually) several[1] equivalent versions of a MA-specification. It is, therefore, common to prefer the (uniquely determined) version that is *invertible*.

DEF (invertibility). We say that the MA(q) process (with $\mu = E(Y_t)$)

(2) $$Y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \cdots + \theta_q\varepsilon_{t-q} = \mu + \theta(L)\varepsilon_t$$

is *invertible* if we can solve $\{\varepsilon_t\}$ with respect to $\{Y_t\}$ (as an "AR(∞)" process)

(3) $$\varepsilon_t = -\psi(1)\mu + Y_t + \psi_1 Y_{t-1} + \psi_2 Y_{t-2} + \cdots = \psi(L)Y_t$$

where $\psi(L) = 1 + \psi_1 L + \psi_2 L^2 + \cdots$ is a proper filter (i.e., satisfying $\sum_j |\psi_j| < \infty$).

This turns out to be possible if and only if all the roots of the companion polynomial of $\theta(L)$, are outside the unit circle (or all roots of the reversed companion polynomial inside). This preference identifies the MA specification. If any of the roots are on the unit circle, the process is not invertible.

Question. Show, either by successive substitution as we did for AR(1) in the first lecture or by other means, that, if $|\theta| < 1$, then $\{Y_t\}$ in (1) is invertible, with AR(∞) solution

$$\varepsilon_t = Y_t + (-\theta)Y_{t-1} + (-\theta)^2 Y_{t-2} + \cdots = \psi(L)Y_t$$

which we can express as

$$\psi(L) = \frac{1}{1+\theta L}$$

[1] as long as some of the roots of the MA lag polynomial do not lie on the unit circle

Exercise 2. On AR(2) processes.

Introduction. Summary of the application of the theory, page 9-17, in the lecture notes for lecture 2 (LN2):

Consider an AR(2) with specification

(4) $$Y_t = \varphi_0 + \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2} + \varepsilon_t \quad \text{where} \quad \varepsilon_t \sim WN(0,\sigma^2)$$

We want to find out if (4) has a causal stationary solution, and, if so, to find it.

First step is to write (4) as $Y_t - \varphi_1 Y_{t-1} - \varphi_2 Y_{t-2} = w_t$, where $w_t = \varphi_0 + \varepsilon_t$, or

(5) $$\varphi(L)Y_t = w_t, \text{ where } \varphi(L) = 1 - \varphi_1 L - \varphi_2 L^2$$

The idea is to try to construct a (causal) linear filter, $\psi(L) = \psi_0 + \psi_1 L + \psi_2 L^2 + \cdots$ such that

(6) $$\psi(L)\varphi(L) = 1,$$

because, then multiplying both sides of (5) by $\psi(L)$ gives us the solution (if it exists) directly (using theorem 2).

$$\psi(L)\varphi(L)Y_t = 1\cdot Y_t = \psi(L)w_t = \psi(L)(\varphi_0 + \varepsilon_t) = \psi(1)\varphi_0 + \psi(L)\varepsilon_t$$

Hence

(7) $$Y_t = \psi(1)\varphi_0 + \psi(L)\varepsilon_t \quad \text{where} \quad \mu = E(Y_t) = \psi(1)\varphi_0$$

Page 10 in LN2 shows that, for $t \geq 2$, the equation $\psi(L)\varphi(L) = 1$ implies that $\{\psi_t\}$ must satisfy the homogeneous version of the difference equation in (5)

$$\psi_t - \varphi_1\psi_{t-1} - \varphi_2\psi_{t-2} = 0, \quad (\text{or } \varphi(L)\psi_t = 0\text{) for } t \geq 2,$$

the general solution of which is (see page 11 of LN2)

$$\psi_t = \begin{cases} c_1 \dfrac{1}{z_1^t} + c_2 \dfrac{1}{z_2^t} = c_1 r_1^t + c_2 r_2^t & \text{if } r_1 \neq r_2 \\ (c_1 + c_2 t) r^t & \text{if } r_1 = r_2 = r \end{cases}$$

where z_1, z_2 are the roots of the companion polynomial to $\varphi(L)$, $\varphi(z) = 1 - \varphi_1 z - \varphi_2 z^2$ and $r_1 = \frac{1}{z_1}$, $r_2 = \frac{1}{z_2}$, the roots of the reversed companion polynomial, $h(x) = x^2 - \varphi_1 x - \varphi_2$, and where c_1, c_2 are arbitrary constants. The initial conditions, $\psi_0 = 1, \quad \psi_1 = \varphi_1$, determine c_1, c_2 to be $c_1 = \frac{\varphi_1 - r_2}{r_1 - r_2}, \quad c_2 = \frac{r_1 - \varphi_1}{r_1 - r_2}$ in case $r_1 \neq r_2$, and $c_1 = 1, \quad c_2 = \frac{\varphi_1}{r} - 1$ in case $r_1 = r_2 = r$.

Now, the only way we can achieve that $\sum_{j=0}^{\infty} |\psi_j| < \infty$, which is the condition that $\psi(L)$ is a proper (causal) filter, is that r_1, r_2 are inside the unit circle (or, equivalently, that z_1, z_2 are outside the unit circle). in which case, $Y_t = \psi(1)\varphi_0 + \psi(L)\varepsilon_t$, is the causal stationary solution.

Now, according to theorem 2, the equation $\psi(L)\varphi(L) = 1$ is equivalent to $\psi(\text{z})\varphi(\text{z}) = 1$ in terms of usual functions, i.e., the companions $\psi(z)$ and $\varphi(z)$, which implies that $\psi(z) = \frac{1}{\varphi(z)}$. From this we define the operator $\frac{1}{\varphi(L)}$ simply to mean the filter $\psi(L)$, which, as we now know, is a well defined causal filter as long as the roots of $\varphi(z)$ are outside the unit circle.

In this way, the solution can be expressed

(8) $\qquad Y_t = \frac{\varphi_0}{\varphi(1)} + \frac{1}{\varphi(L)}\varepsilon_t$, where $\mu = E(Y_t) = \frac{\varphi_0}{\varphi(1)} = \frac{\varphi_0}{1 - \varphi_1 - \varphi_2}$

Question A. The polynomial , $4 + (z-1)^2 = z^2 - 2z + 5$, we used in the appendix to LN2 to motivate complex numbers, has two complex roots, $1 + 2i$ and $1 - 2i$. Show that

$$\frac{1}{1+2i} = \frac{1}{5} - \frac{2}{5}i \quad \text{(\textbf{Hint:} multiply both denominator and numerator by } 1-2i\text{)}$$

and

$$\frac{1}{1-2i} = \frac{1}{5} + \frac{2}{5}i$$

Which of the two pairs, $1 \pm 2i$ and $\frac{1}{5} \pm \frac{2}{5}i$, are inside the unit circle, and which are outside?

B. Which of the two polynomials, $z^2 - 2z + 5$, or the reversed one obtained by putting $z = 1/x$, i.e., $5x^2 - 2x + 1$, could serve as the companion polynomial to the AR(2) process in (4), $Y_t = \varphi_0 + \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2} + \varepsilon_t$ where $\varepsilon_t \sim WN(0, \sigma^2)$, and where we have put $\varphi_0 = 2$, in order that $\{Y_t\}$ becomes a causal AR(2) process?
Identify φ_1 and φ_2.
Determine the solution of the process.
Find the long run effect of a unit change in ε_t (keeping all other ε_s's untouched).
Find $E(Y_t)$.

C. Using any software of your convenience, simulate Y_t in **B** for $t = 0,1,2,\ldots,100$, assuming $\{\varepsilon_t\}$ are iid and standard normal ($\varepsilon_t \sim N(0,1)$). Make a time series plot of the result.
Make also a plot of the series, $\{\psi_t\}$, of dynamic multipliers. Looking at the plot, how long time does it take (roughly) before the ψ_t's become negligible?
[**Hint.** In the appendix I have done all this, using Stata, for the example on page 14 in LN2. Your job is just to copy that procedure for this example.]

Note. Causal ARMA(*p*,*q*) processes are often called "*short-memory*" processes since ψ_t approaches 0 exponentially fast (as r_i^t for $|r_i| < 1$) for such processes. An example of a "*long-memory*" process would be $Y_t = \psi(L)\varepsilon_t$, where, e.g., $\psi_t = 1/t^2$. Such a filter is proper and causal, since (from general math) $\sum_{t=1}^{\infty} |\psi_j| = \sum_{t=1}^{\infty} 1/t^2 = \pi^2/6 < \infty$, but ψ_t approaches 0 slower than exponentially.

Exercise 3.
A. The ARMA(*p*,*q*) difference equation (*) $\varphi(L)Y_t = \varphi_0 + \theta(L)\varepsilon_t$ has a causal stationary solution, $Y_t = \mu + \frac{\theta(L)}{\varphi(L)}\varepsilon_t$ if the roots of $\varphi(z)$ are outside the unit circle, and where $\mu = E(Y_t) = \frac{\varphi_0}{\varphi(1)}$. An easy way to get rid of the constant φ_0 in (*) is to center the Y_t -series: Introduce, $y_t = Y_t - \mu$, and show that (**) $\varphi(L)\,y_t = \theta(L)\varepsilon_t$ without the constant. [**Hint.** Replace y_t by $Y_t - \mu$ in (**) and use the bullet points in (9), page 5 in LN2.]

B. **i)** The general condition that (*) has a causal stationary solution is that all the roots of the companion polynomial $\varphi(z)$ lie outside the unit circle (or equivalently that all roots of the reversed companion polynomial lie inside the unit circle). If **(ii)** some roots are inside and some outside the unit circle and no roots lie on the unit circle, a stationary but non-causal solution exists. If **(iii)** at least one root lies *on* the unit circle, then no stationary solution exists.

To illustrate **(iii),** suppose the lag polynomial for Y_t is

$$\varphi(L) = 1 - 2.3L + 1.7L^2 - 0.4L^3 = (1-L)(1-0.5L)(1+0.8L)$$

factorized along the lines on top of page 12 in LN2. Identify which bullet rule(s) from properties (9), page 5 in LN2, I use in the following argument:
We have

$$\varphi(L)Y_t = (1-L)(1-0.5L)(1+0.8L)Y_t =$$
$$= (1-0.5L)(1+0.8L)(1-L)Y_t = (1-0.5L)(1+0.8L)\Delta Y_t$$

implying that $u_t = \Delta Y_t$ is a causal stationary process. Solving the last equation with respect to Y_t starting in $t = -1$ gives $Y_t = \Delta Y_0 + u_1 + u_2 + \cdots + u_t$, which, in general, is a non-stationary random walk type of process (the last point you don't need to justify).

C. Explore the status (**i., ii., or iii**.) of the following 4 AR models (taken from Ragnar's first-seminar exercises 2011). If necessary, you may utilize the `polyroots` command in Stata as in the LN2 appendix (or other software):

(a) $Y_t = 1.5Y_{t-1} - 0.5Y_{t-2} + \varepsilon_t$

(b) $Y_t = Y_{t-2} + \varepsilon_t$

(c) $Y_t = 2Y_{t-1} - Y_{t-2} + \varepsilon_t$

(d) $Y_t = Y_{t-1} + 0.25Y_{t-2} - 0.25Y_{t-3} + \varepsilon_t$

Appendix to exercise 2.

The model on LN2 page 14 is

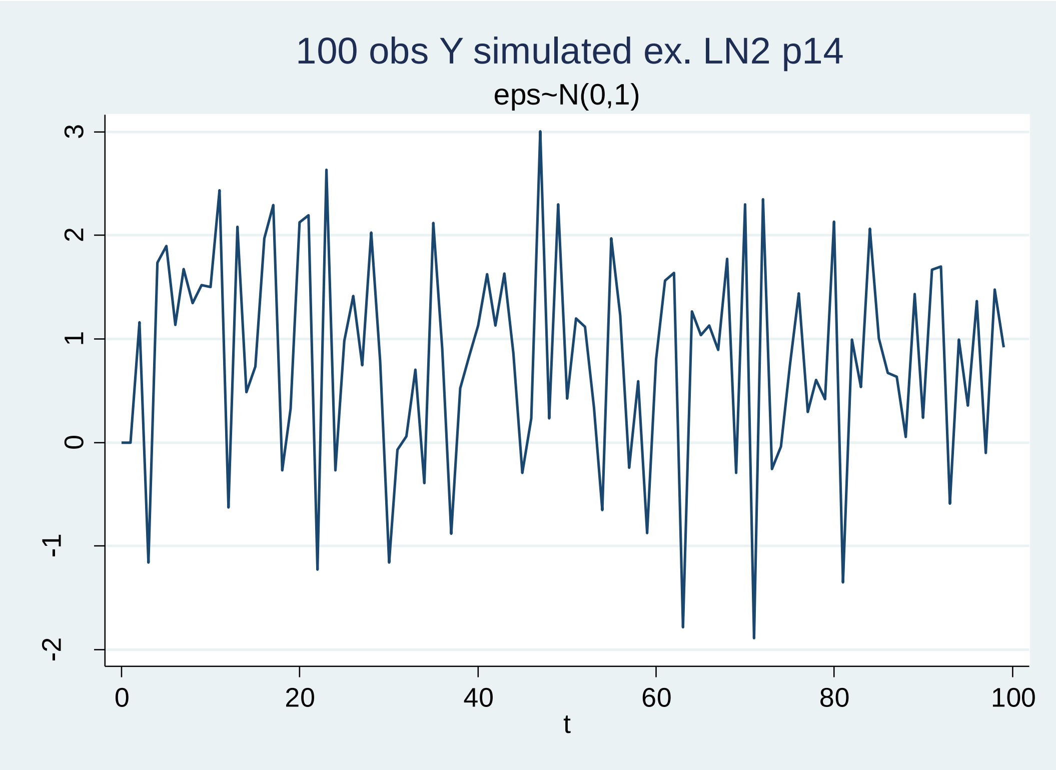
(*) $Y_t = 1 - 0.3Y_{t-1} + 0.1Y_{t-2} + \varepsilon_t$ where I here assume $\{\varepsilon_t\}$ are iid with $\varepsilon_t \sim N(0,1)$

Stata: (I use the `forecast`-command documented under time-series. Read first the pdf-documentation under `forecast coefvector`)

First simulate the Y's

```
. set obs 100
obs was 0, now 100

. gen y=0                                (starts with a column of 0)

. gen t=_n-1                             (generate time t=0,1,… 99)

. tsset t                                (initiate t as a neutral time-series
        time variable:  t, 0 to 99       var, in order to get access to time-
```

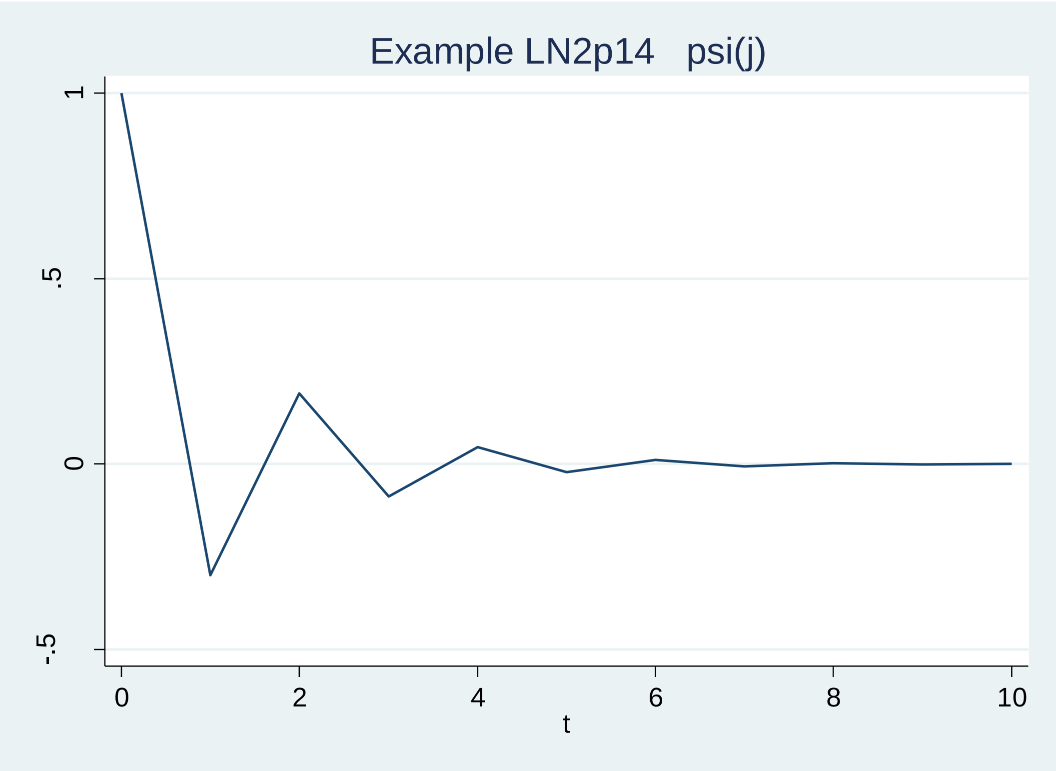
```
                delta:  1 unit                series functions, commands and plots)

. gen eps=rnormal()                          (generate White noise)

. matrix y=(1,-.3,.1,1)                      (make vector of coefficients in (*).
                                             The matrix brings you into mata in
                                             one step and immediately out of mata)

. matrix coleq y=y:_cons y:L.y y:L2.y y:eps   (Define model, one equation called y.
                                             L.y means y lagged one, L2.y means y
                                             lagged twice. in addition will eps be
                                             added to the y-equation)

. matrix list y                              (To control my model)

y[1,4]
        y:      y:      y:      y:
                L.     L2.
    _cons       y       y     eps
r1      1     -.3      .1       1

. forecast create                            (initiate new model)
  Forecast model started.

. forecast coefvector y                      (attach coefficients to y)
  Forecast model now contains 1 endogenous variable.

. forecast solve, begin(2) log(off)          (calculate all y's from model and eps,
                                             starting from t=2, implying 2
                                             start values =0. The answer comes in
                                             a new column called f_y

Computing dynamic forecasts for current model.
----------------------------------------------
Starting period:  2
Ending period:    99
Forecast prefix:  f_

Forecast 1 variable spanning 98 periods.
----------------------------------------

. tsline f_y                                 (plot the time series)
```



Now generate the dynamic multipliers ψ_t

```
. gen psi=0

. matrix psi=(-.3,.1)                    (define coef. vector for psi diff. eq)

. replace psi = 1 in 1                   (define start values psi(0)=1,
                                         psi(1)=fi(1)=-0.3 in the edit window)
(1 real change made)

. replace psi = -.3 in 2
(1 real change made)

. forecast create, replace               (initiate new model)
  (Forecast model ended.)
  Forecast model started.

. matrix coleq psi= psi:L.psi psi:L2.psi  (define variables for the coef.'s)

. matrix list psi

psi[1,2]
     psi:  psi:
       L.   L2.
     psi   psi
r1   -.3    .1

. forecast coefvector psi                            (initiate full psi-model)
  Forecast model now contains 1 endogenous variable.

. forecast solve, begin(2) log(off)                  (calculate all psi(t)'s
                                                     answer in column f_psi)

Computing dynamic forecasts for current model.
----------------------------------------------
Starting period:  2
```

```
Ending period:     99
Forecast prefix:   f_

Forecast 1 variable spanning 98 periods.
----------------------------------------


. tsline f_psi if t<=10       (the plot of the first 10 psi's is sufficient here!)
```



The multipliers seem to have an effect for about 5-6 time units in this example.