HG
Second lecture - 22. Jan. 2014

1. Dynamic Multipliers.

Suppose $Y_t \sim AR(1)$ which is a dynamic model for the time series, $\{Y_t\}$, where the dynamics is specified in terms of a stochastic difference equation

(1) $$Y_t = \varphi_0 + \varphi_1 Y_{t-1} + \varepsilon_t \quad \text{where}$$
$$Y_t = \mu + Z_t \quad \text{where}$$
$$Z_t = \sum_{j=-\infty}^{\infty} \psi_{-j}\varepsilon_{t+j} = \cdots + \psi_2\varepsilon_{t-2} + \psi_1\varepsilon_{t-1} + \psi_0\varepsilon_t + \psi_{-1}\varepsilon_{t+1} + \psi_{-2}\varepsilon_{t+2} + \cdots$$
$$\varepsilon_t \sim WN(0,\sigma^2)$$

Remember:

i. If $|\varphi_1| < 1$, (1) has a **causal** stationary solution

(2) $$Y_t = \frac{\varphi_0}{1-\varphi_1} + \varepsilon_t + \varphi_1\varepsilon_{t-1} + \varphi_1^2\varepsilon_{t-2} + \cdots + \varphi_1^t\varepsilon_0 + \varphi_1^{t+1}\varepsilon_{-1} + \cdots$$

ii. If $|\varphi_1| > 1$, (1) has a **non-causal** stationary solution

$$Y_t = \frac{\varphi_0}{1-\varphi_1} + \eta_{t+1} + b_1\eta_{t+2} + b_1^2\eta_{t+3} + \cdots + b_1^j\eta_{t+j+1} + \cdots$$

where $\eta_{t+1} = -\frac{\varepsilon_{t+1}}{\varphi_1} \sim WN\left(0,\ \frac{\sigma^2}{\varphi_1^2}\right)$. Any causal solution is non-stationary and explosive.

iii. If $|\varphi_1| = 1$, (1) has no stationary solution. Its solution is a random walk plus (if $\varphi_0 \neq 0$) a deterministic trend.

A **general stationary solution** for a dynamically modelled univariate time series, $\{Y_t\}$ - with only one white noise error term, ε_t - has the form

(3)
$$Y_t = \mu + Z_t \quad \text{where}$$
$$Z_t = \sum_{j=-\infty}^{\infty} \psi_{-j}\varepsilon_{t+j} = \cdots + \psi_2\varepsilon_{t-2} + \psi_1\varepsilon_{t-1} + \psi_0\varepsilon_t + \psi_{-1}\varepsilon_{t+1} + \psi_{-2}\varepsilon_{t+2} + \cdots$$

where $\varepsilon_t \sim WN(0,\sigma^2)$, and $\{\psi_j\}_{j=-\infty}^{\infty}$ is a sequence of numbers satisfying

(4)
$$\sum_{j=-\infty}^{\infty} \left|\psi_j\right| < \infty$$

The condition (4) ensures that Z_t is a meaningful random variable with expectation 0 ($\Rightarrow E(Y_t) = \mu$), and covariance stationary with autocovariance ,

(5)
$$\gamma(-h) = \gamma(h) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_{j+h}\psi_j$$

A solution is **causal** - which we will concentrate on - when all $\psi_{-1}, \psi_{-2}, \ldots$ are 0, so that

(6)
$$Y_t = \mu + Z_t \quad \text{where}$$
$$Z_t = \sum_{j=0}^{\infty} \psi_j\varepsilon_{t+j} = \quad \psi_0\varepsilon_t + \psi_1\varepsilon_{t-1} + \psi_2\varepsilon_{t-2} + \cdots$$

where Z_t is covariance stationary with $E(Z_t) = 0$ and

(7)
$$\gamma(-h) = \gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_{j+h}\psi_j$$

E.g., in the (stable) AR(1) case, $\psi_j = \varphi_1^j$ (with $\psi_0 = 1$ and $|\varphi_1| < 1$), so the covariance function for Z_t (and for Y_t as well) is

$$\gamma(-h) = \gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_{j+h}\psi_j = \sigma^2 \sum_{j=0}^{\infty} \varphi_1^{2j+h}$$

$$= \sigma^2 \varphi_1^h \left[1 + \varphi_1^2 + \left(\varphi_1^2\right)^2 + \left(\varphi_1^2\right)^3 + \cdots\right] = \sigma^2 \frac{\varphi_1^h}{1-\varphi_1^2}$$

Interpretation of the ψ_j 's

The ψ_j **'s** in the linear process solution (7) are called **dynamic multipliers** in the time series $\{\mathrm{Y}_t\}$, i.e., the effect of a unit change in ε_t on $\{\mathrm{Y}_{t+j}\}_{j=0}^{\infty}$, when there are no changes in the other ε_s 's. Due to the linear structure, this effect can be obtained by the derivative, $\partial Y_{t+j} / \partial \varepsilon_t$:

$$\frac{\partial \mathrm{Y}_{t+j}}{\partial \varepsilon_t} = \frac{\partial}{\partial \varepsilon_t}\left[\mu + \psi_0 \varepsilon_{t+j} + \psi_1 \varepsilon_{t+j-1} + \psi_2 \varepsilon_{t+j-2} + \cdots + \psi_j \varepsilon_t + \cdots\right] = \psi_j$$

for $j = 0, 1, 2, \dots$.

The immediate effect on Y_t is $\psi_0 = 1$ (usually).

The effect on Y_{t+1} is ψ_1

….

The effect on Y_{t+j} is ψ_j

which $\to 0$ as $j \to \infty$ since $\sum_j |\psi_j| < \infty$.

The total effect on $\{\mathrm{Y}_t\}$ of a unit change in ε_t (i.e., "**the long run effect**") is then given as

(8) $$\sum_{j=0}^{\infty} \frac{\partial Y_{t+j}}{\partial \varepsilon_t} = \sum_{j=0}^{\infty} \psi_j$$

In the stable AR(1) case, the long run effect becomes

$$\sum_{j=0}^{\infty}\frac{\partial Y_{t+j}}{\partial \varepsilon_t}=\sum_{j=0}^{\infty}\varphi_1^j=\frac{1}{1-\varphi_1}$$

Sometimes, when Y_t is measured in money terms, one may be interested in the *discounted* long run effect (discount factor β), defined by

$$\sum_{j=0}^{\infty}\beta^j\frac{\partial Y_{t+j}}{\partial \varepsilon_t}=\sum_{j=0}^{\infty}\beta^j\psi_j\overset{\text{stable AR(1) case}}{=}\sum_{j=0}^{\infty}\beta^j\varphi_1^j=\frac{1}{1-\beta\varphi_1}$$

Similar simple expression can be derived in the general stable AR(p) case.

2. Lag Polynomials and Linear Filters

We now introduce the lag-operator, L, working on a time series $\{Y_t\}$, and defined by, $LY_t \overset{def}{=} Y_{t-1}$. Similarly we define, $L^2 = L\cdot L$ by $L^2Y_t = L(LY_t) = L(Y_{t-1}) = Y_{t-2}$, and in general, $L^jY_t = Y_{t-j}$, $j = 0,1,2,\ldots$ (implicitly defining, $L^0Y_t = Y_t$)

We extend this definition to the forward shift operator, L^{-1}, defined by $L^{-1}Y_t = Y_{t+1}$, so that $L^{-1}LY_t = L^{-1}(LY_t) = \mathrm{L}^{-1}Y_{t-1} = Y_t$ (similarly $LL^{-1}Y_t = Y_t$).

Hence, $LL^{-1} = L^{-1}L = L^0$.

We get as above, $L^{-j}Y_t = Y_{t+j}$, $j = 0,1,2,\ldots$

Using L as basis operator, we can build up quite general operators. For example, we had the first difference operator, Δ, defined by $\Delta Y_t = Y_t - Y_{t-1} = Y_t - LY_t$. The last expression we write as $(1-L)Y_t$ *by definition*, where we consider $1-L$ a simple lag-polynomial. We identify this with Δ, i.e., $\Delta = 1-L$.

The second difference, as a lag polynomial becomes

$$\Delta^2 Y_t = \Delta(\Delta Y_t) = \Delta(Y_t - Y_{t-1}) = Y_t - Y_{t-1} - (Y_{t-1} - Y_{t-2}) = \mathrm{Y}_t - 2Y_{t-1} + Y_{t-2} \overset{\text{def}}{=}$$
$$= (1 - 2L + L^2)Y_t$$

We see that we get the same answer if we multiply out Δ^2 as if *L* was just a number:

$$\Delta^2 = (1-L)^2 = 1 - 2L + L^2$$

We now define a general lag-polynomial of order *p* as the operator

$$\alpha(L) = \alpha_0 + \alpha_1 L + \alpha_2 L^2 + \cdots + \alpha_p L^p$$

that, by definition, works on any series $\{\mathrm{w}_t\}$ (stochastic or not)

$$\alpha(L)w_t = \left(\alpha_0 + \alpha_1 L + \alpha_2 L^2 + \cdots + \alpha_p L^p\right) w_t \overset{\text{def}}{=}$$
$$= \alpha_0 w_t + \alpha_1 L w_t + \alpha_2 L^2 w_t + \cdots + \alpha_p L^p w_t =$$
$$= \alpha_0 w_t + \alpha_1 w_{t-1} + \alpha_2 w_{t-2} + \cdots + \alpha_p w_{t-p}$$

The operator $\alpha(L)$ corresponds to a usual polynomial (*the companion polynomial*), $\alpha(\mathrm{z}) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_p z^p$, where z is a numeric variable.

(9) Some much used properties:

- $\alpha(L)(u_t + w_t) = \alpha(L)u_t + \alpha(L)w_t$ (follows directly from the definition – just write it out).
- If $w_t = c$ is constant for all *t*, clearly $Lw_t = Lc = c$ and $L^j w_t = L^j c = c$, and

  $$\alpha(L)c = \alpha_0 c + \alpha_1 Lc + \alpha_2 L^2 c + \cdots + \alpha_p L^p c =$$
  $$= (\alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_p)c = \alpha(1)c$$

  where $\alpha(1) = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_p$ is obtained by replacing *z* by 1 in the companion polynomial.

- If $\alpha(L)=\alpha_0+\alpha_1 L+\cdots+\alpha_p L^p, \quad \beta(L)=\beta_0+\beta_1 L+\cdots+\beta_q L^q$ are two lag polynomials, the product $\alpha(L)\beta(L)$ is commutative (the order of multiplication does not matter) , and a lag polynomial of order (*p*+*q*), $\alpha(L)\beta(L)=\beta(L)\alpha(L)=\delta_0+\delta_1 L+\cdots+\delta_{p+q}L^{p+q}$, where the coefficients, δ_j , can be found by multiplying the corresponding companion polynomials in the usual manner, $\alpha(\mathrm{z})\beta(\mathrm{z})=\beta(\mathrm{z})\alpha(\mathrm{z})=\delta_0+\delta_1\mathrm{z}+\cdots+\delta_{p+q}z^{p+q}$ (straightforward, although somewhat messy to prove…)

Unfortunately, we need to go one step further, and define lag "polynomials" of infinite orders. A causal stationary solution for $\{\mathrm{Y}_t\}$, expressed by *L*, has the form from (6)

$$
\begin{aligned}
Y_t-\mu &= \sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j} = \psi_0\varepsilon_t+\psi_1\varepsilon_{t-1}+\psi_2\varepsilon_{t-2}+\cdots = \\
&=\psi_0\varepsilon_t+\psi_1 L\varepsilon_t+\cdots+\psi_j L^j\varepsilon_t+\cdots \overset{\text{def}}{=} \left(\sum_{j=0}^{\infty}\psi_j L^j\right)\varepsilon_t \\
&=\psi(L)\varepsilon_t
\end{aligned}
$$

where $\psi(L)=\sum_{j=0}^{\infty}\psi_j L^j$, satisfying $\sum_j|\psi_j|<\infty$, thus, is an "infinite order lag polynomial", with the more common name, **linear (time-homogeneous) filter,** a basic tool in time series analysis.

A special case is the MA(*q*) process

$$
Y_t-\mu=\sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j} = \varepsilon_t+\theta_1\varepsilon_{t-1}+\theta_2\varepsilon_{t-2}+\cdots+\theta_q\varepsilon_{t-q}=\theta(L)\varepsilon_t
$$

where $\psi_j=0$ for $j>q, \quad \psi_j=\theta_j$ for $j\le q,$ and $\psi_0=1,$ and

$$
\theta(L)=1+\theta_1 L+\theta_2 L^2+\cdots+\theta_q L^q
$$

For such linear filters strong mathematical results can be proven:

Theorem 1. If $\psi(L)$ is a linear filter satisfying $\sum_j |\psi_j| < \infty$, and $\{u_t\}$ is any covariance stationary time series, then the filtered series

$$Y_t = \psi(L)u_t = \psi_0 u_t + \psi_1 u_{t-1} + \psi_2 u_{t-2} + \cdots + \psi_j u_{t-j} + \cdots$$

are well defined as random variables and the series $\{Y_t\}$ is covariance stationary.

(The proof requires some Hilbert space theory and is skipped here. Formulaes for the autocovariance can be found in other textbooks, e.g., in Brockwell and Davis referred to below.)

Also the three properties in (9) are still valid. First we define the companion infinite order "polynomial", called a **power series** where *z* is a usual numeric variable:

$$\psi(\mathrm{z}) = \psi_0 + \psi_1 \mathrm{z} + \psi_2 \mathrm{z}^2 + \cdots + \psi_j \mathrm{z}^j + \cdots$$

Corresponding to (9) we then have (formulated for causal linear filters that we need here, but is valid also for more general and non-causal filters):

Theorem 2. If $\psi(L)$ is a (causal) linear filter satisfying $\sum_j |\psi_j| < \infty$, the following is true:

i. $\psi(L)(u_t + w_t) = \psi(L)u_t + \psi(L)w_t$ (i.e., $\psi(L)$ is a linear operator)

ii. If $w_t = c$ is constant for all *t*,

$$\psi(L)c = \psi_0 c + \psi_1 Lc + \psi_2 L^2 c + \cdots =$$
$$= (\psi_0 + \psi_1 + \psi_2 + \cdots)c = \psi(1)c$$

where $\psi(1) = \psi_0 + \psi_1 + \psi_2 + \cdots = \sum_{j=0}^{\infty} \psi_j$ is obtained by replacing *z* by 1 in the companion power series.

iii. If $\alpha(L) = \alpha_0 + \alpha_1 L + \alpha_2 L^2 + \cdots, \quad \beta(L) = \beta_0 + \beta_1 L + \beta_2 L^2 + \cdots$ are two linear filters (satisfying finite sum of absolute values of coefficients), the product $\alpha(L)\beta(L)$ is commutative, and defines a linear filter

$\alpha(L)\beta(L) = \beta(L)\alpha(L) = \delta_0 + \delta_1 L + \delta_2 L^2 + \cdots$, where $\sum_j |\delta_j| < \infty$,

and where the coefficients, δ_j, can be found by multiplying the corresponding companion power series in the usual manner,

$$\alpha(\text{z})\beta(\text{z}) = \beta(\text{z})\alpha(\text{z}) = \delta_0 + \delta_1 \text{z} + \delta_2 \text{z}^2 + \cdots$$

(The proof of **i.** and **ii.** is straightforward as in (9). The proof of **iii.** requires some theory of complex power series that we skip here. Formal proofs can be found, e.g., in P.J. Brockwell and R.A. Davis, “Time Series: Theory and Methods”, *Springer Verlag* 1987 or later.)

Application.
We are here primarily concerned with (possible) causal solutions of

(10) AR(p) with specification

$$Y_t = \varphi_0 + \varphi_1 Y_{t-1} + \cdots + \varphi_p Y_{t-p} + \varepsilon_t \quad \text{where} \quad \varepsilon_t \sim WN(0,\sigma^2)$$

and

(11) ARMA(p,q) with specification

$$Y_t = \varphi_0 + \varphi_1 Y_{t-1} + \cdots + \varphi_p Y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}, \text{ where}$$

$$\varepsilon_t \sim WN(0,\sigma^2)$$

As a first step we write (10) and (11) as

$$Y_t - \varphi_1 Y_{t-1} - \cdots - \varphi_p Y_{t-p} = w_t, \quad \text{or, with corresponding lag polynomial}$$

(12)	$\varphi(L)Y_t = w_t$, where $\varphi(L) = 1 - \varphi_1 L - \varphi_2 L^2 - \cdots - \varphi_p L^p$

and where $w_t = \varphi_0 + \varepsilon_t$ in the AR(p) case, and
$w_t = \varphi_0 + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} = \varphi_0 + \theta(L)\varepsilon_t$ in the ARMA(p,q) case.

The idea is to try to construct a (causal) linear filter,
$\psi(L) = \psi_0 + \psi_1 L + \psi_2 L^2 + \cdots$ such that

(13)	$\psi(L)\varphi(L) = 1$

If we succeed with that, we can simply multiply both sides of (12) to obtain our solution:

$$\psi(L)\varphi(L)Y_t = 1 \cdot Y_t = \psi(L)w_t \quad \text{giving (using theorem 2)}$$

AR(p) case: $Y_t = \psi(L)(\varphi_0 + \varepsilon_t) \overset{\text{i.}}{=} \psi(L)\varphi_0 + \psi(L)\varepsilon_t \overset{\text{ii.}}{=} \psi(1)\varphi_0 + \psi(L)\varepsilon_t$
Hence, solving (13) gives us the directly the linear filter solution.

The ARMA(p,q) case is a little more complicated. Multiplying both sides by $\psi(L)$, we get

$$Y_t = \psi(L)(\varphi_0 + \theta(L)\varepsilon_t) = \psi(1)\varphi_0 + \psi(L)\theta(\mathrm{L})\varepsilon_t$$

We see that, in this case, the job is not completely done yet since we need to write the linear filter $\psi(L)\theta(\mathrm{L}) = \delta(L) = 1 + \delta_1 L + \delta_2 L^2 + \delta_3 L^3 + \cdots$ in the form of a linear filter, $\delta(L) = 1 + \delta_1 L + \delta_2 L^2 + \delta_3 L^3 + \cdots$. This is best accomplished by the method of solving a difference equation as described in the summary at the end before the appendix.

Note, however, that in both cases, we find $\mu = E(Y_t) = \psi(1)\varphi_0 = \left(\sum_{j=0}^{\infty} \psi_j\right) \cdot \varphi_0$

Now to (13) relevant for the AR(p) case: To simplify the notation we look at the case, $p = 2$ (the general case is quite similar):

$$
\begin{aligned}
(13) \Leftrightarrow \quad 1 &= (1 - \varphi_1 L - \varphi_2 L^2)(\psi_0 + \psi_1 L + \psi_2 L^2 + \cdots) = \\
&= \psi_0 + \psi_1 L + \quad \psi_2 L^2 + \quad \psi_3 L^3 + \quad \psi_4 L^4 + \cdots \\
&\quad - \varphi_1 \psi_0 L - \varphi_1 \psi_1 L^2 - \varphi_1 \psi_2 L^3 - \varphi_1 \psi_3 L^4 - \cdots \\
&\qquad\qquad - \varphi_2 \psi_0 L^2 - \varphi_2 \psi_1 L^3 - \varphi_2 \psi_2 L^4 - \cdots
\end{aligned}
$$

Collecting terms we get

$$1 = \psi_0 + (\psi_1 - \varphi_1\psi_0)L + (\psi_2 - \varphi_1\psi_1 - \varphi_2\psi_0)L^2 + \cdots + (\psi_j - \varphi_1\psi_{j-1} - \varphi_2\psi_{j-2})L^j + \cdots$$

For this equality to hold, we must have $\psi_0 = 1$ and all coefficients for any term with some L^k, $k = 1, 2, \ldots$, must be equal to 0. Hence

$\psi_0 = 1, \quad \psi_1 = \varphi_1$, and for all $j \geq 2$, we must have $\psi_j - \varphi_1\psi_{j-1} - \varphi_2\psi_{j-2} = 0$.

In other words: The sequence $\{\psi_t\}$ (for $t \geq 2$) must satisfy *the same* difference equation as $\{Y_t\}$ in (12) only with 0 instead of w_t on the right side. Such difference equations (with 0 on the right side) are called *homogeneous*.

So, the task has been reduced to finding the general solution of the homogeneous difference equation

(14) $\psi_t - \varphi_1\psi_{t-1} - \varphi_2\psi_{t-2} = 0$, (or $\varphi(L)\psi_t = 0$) for $t \geq 2$.

Note that also the autocovariance function $\{\gamma(h)\}$ satisfies the same difference equation for $h \geq 2$ (see next lecture or Hamilton).

We find the general solution (see e.g., Sydsæter II) from the companion polynomial and its roots, i.e., the solutions, z_1, z_2 of

(15) $1 - \varphi_1 z - \varphi_2 z^2 = 0$

Hamilton operates with the roots, r_1, r_2, of the "reversed" version of the polynomial

(16) $x^2 - \varphi_1 x - \varphi_2 = (x - r_1)(x - r_2) = 0$

Clearly, we must have $r_i = 1/z_i$, $i = 1,2$

[for putting $z = 1/x$ in (15) gives, $1 - \varphi_1 z - \varphi_2 z^2 = \frac{1}{x^2}(x^2 - \varphi_1 x - \varphi_2)$, which is 0 for $x = r_1$ or r_2, i.e., for $z = 1/r_1$ or $1/r_2$ implying $z_i = 1/r_i$, $i = 1,2$]

Factorizing (15), we get (writing $z = 1/x$)

$$1 - \varphi_1 z - \varphi_2 z^2 = \frac{1}{x^2}(x^2 - \varphi_1 x - \varphi_2) = \frac{1}{x^2}(x - r_1)(x - r_2) =$$
$$= z^2\left(\frac{1}{z} - r_1\right)\left(\frac{1}{z} - r_2\right) = (1 - r_1 z)(1 - r_2 z)$$

Hence, the corresponding lag polynomial can be factorized

$$1-\varphi_1 L-\varphi_2 L^2=(1-r_1 L)(1-r_2 L)=\left(1-\frac{1}{z_1}L\right)\left(1-\frac{1}{z_2}L\right)$$

The case of different roots. If $z_1 \neq z_2$ (i.e., $r_1 \neq r_2$), the general solution of (14) is

(17) $\psi_t = c_1\frac{1}{z_1^t}+c_2\frac{1}{z_2^t}=c_1 r_1^t + c_2 r_2^t$ where c_1, c_2 are arbitrary constants

To see this we substitute in (14) (for $t \geq 2$)

$$\begin{aligned}\psi_t-\varphi_1\psi_{t-1}-\varphi_2\psi_{t-2} &= c_1 r_1^t + c_2 r_2^t-\varphi_1(c_1 r_1^{t-1}+c_2 r_2^{t-1})-\varphi_2(c_1 r_1^{t-2}+c_2 r_2^{t-2})= \\ &= c_1(r_1^t-\varphi_1 r_1^{t-1}-\varphi_2 r_1^{t-2})+c_2(r_2^t-\varphi_1 r_2^{t-1}-\varphi_2 r_2^{t-2})= \\ &= c_1 r_1^{t-2}(r_1^2-\varphi_1 r_1-\varphi_2)+c_2 r_2^{t-2}(r_2^2-\varphi_1 r_2-\varphi_2)=0\end{aligned}$$

Now, we further determine the free constants c_1, c_2 such that our two initial conditions, $\psi_0 = 1, \quad \psi_1 = \varphi_1$, are fulfilled, i.e.,

(18) $$\begin{aligned} 1 &= \psi_0 = c_1 r_1^0 + c_2 r_2^0 = c_1 + c_2 \\ \varphi_1 &= \psi_1 = c_1 r_1^1 + c_2 r_2^1 = c_1 r_1 + c_2 r_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= \frac{\varphi_1 - r_2}{r_1 - r_2} \\ c_2 &= \frac{r_1 - \varphi_1}{r_1 - r_2} \end{aligned}$$

With these values of c_1, c_2 the solution is

(19) $\psi_t = c_1\frac{1}{z_1^t}+c_2\frac{1}{z_2^t}=c_1 r_1^t + c_2 r_2^t$ for $t = 0,1,2,3,\ldots$

From this we get **the stability condition** (i.e., the same as the condition that the solution time series is a causal linear filter (see note next page)):

The condition that the solution filter $\psi(L)=\psi_0+\psi_1 L+\psi_2 L^2+\cdots$ is a proper filter is that

$$\sum_{j=0}^{\infty}|\psi_j|=\sum_{j=0}^{\infty}|c_1 r_1^t+c_2 r_2^t|<\infty$$

The necessary and sufficient condition for this is that both roots must be less than 1 in absolute value ($|r_i|<1$), otherwise the sum must be ∞ . The roots are sometimes complex (see appendix), so this is the same as saying that the roots of the reversed companion polynomial must be inside the unit circle (or, equivalently, the roots of the companion polynomial must be outside the unit circle, $|z_i|>1$).

Note on stability. It is reasonable to call this root condition **a stability condition,** since, not only will the condition ensure the existence of a causal stationary solution, $Y_t=\psi(L)w_t$, but assuming that $\{Y_t\}$ starts somewhere (arbitrary) in *p* (fixed) start values, $Y_0,Y_{-1},Y_{-2},\ldots,Y_{-(p-1)}$, a similar argument as for the AR(1) case will show that the resulting series is not necessarily stationary from the start, but it will approximate the stationary solution as time *t* increases. I.e., the stationary solution appears to play the role as an equilibrium state for $\{Y_t\}$. It is also possible to establish a special start condition by postulating that the start variables, $Y_0,Y_{-1},Y_{-2},\ldots,Y_{-(p-1)}$ have the particular joint distribution determined by the infinite solution filter $\psi(L)$. Then $\{Y_t\}$ will be stationary from the start.

The case of equal roots ($r_1=r_2=r$) is similar but the general solution of (14)

$$\psi_t-\varphi_1\psi_{t-1}-\varphi_2\psi_{t-2}=0, \quad \text{(or } \varphi(L)\psi_t=0 \text{) for } t\ge 2.$$

looks different:

(20) $\psi_t=(c_1+c_2 t)r^t$ for $t\ge 2$ and c_1,c_2 arbitrary constants that will be

determined by the initial conditions $\psi_0 = 1, \quad \psi_1 = \varphi_1$:

$$\begin{aligned} 1 = \psi_0 &= (c_1 + c_2 0)r^0 = c_1 \\ \varphi_1 = \psi_1 &= (c_1 + c_2 1)r^1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 1 \\ c_2 &= \frac{\varphi_1}{r} - 1 \end{aligned}$$

The condition for a causal linear filter solution and stability is also here that the common root, *r*, is inside the unit circle ($|r| < 1$). (Then, and only then,

$\sum_{j=0}^{\infty} |\psi_j| < 1$)

A numerical example.

Consider the following AR(2) specification

(*) $\quad Y_t = 1 - 0.3Y_{t-1} + 0.1Y_{t-2} + \varepsilon_t \quad$ where $\quad \varepsilon_t \sim WN(0, \sigma^2)$

Does (*) have a causal stationary solution?
Is $\{Y_t\}$ stable?
Find the causal solution, the long-run effect, and $E(Y_t)$.

Write first (*) as (12): $\quad Y_t + 0.3Y_{t-1} - 0.1Y_{t-2} = 1 + \varepsilon_t = w_t$

Lag polynomial: $\quad \varphi(L) = 1 + 0.3L - 0.1L^2$

$$\Rightarrow \varphi_0 = 1, \ \varphi_1 = -0.3, \ \varphi_2 = 0.1$$

Companion polynomial: $\quad \varphi(z) = 1 + 0.3z - 0.1z^2$

Reversed comp. polynomial: $\quad h(x) = x^2 + 0.3x - 0.1$

Roots, i.e., solutions of $h(x)=0$:

$$r_i=\frac{1}{2}\left(-0.3\pm\sqrt{0.09+0.4}\right)=\frac{1}{2}\left(-0.3\pm 0.7\right)=\begin{cases}0.2\\-0.5\end{cases}$$

Both roots are less than 1 in absolute value, so the answer to the first two questions is “Yes”, a causal stationary solution exists and the system is stable (implying convergence to the stationary solution for $\{Y_t\}$ from any start-value).

The solution filter, $\psi(L)=\sum_{j=0}^{\infty}\psi_j L^j=\sum_{j=0}^{\infty}(c_1r_1^j+c_2r_2^j)L^j$, where

$$c_1=\frac{\varphi_1-r_2}{r_1-r_2}=\frac{-0.3+0.5}{0.7}=\frac{2}{7}$$

$$c_2=\frac{r_1-\varphi_1}{r_1-r_2}=\frac{0.2+0.3}{0.7}=\frac{5}{7}$$

$$\psi(L)=\sum_{j=0}^{\infty}\left(\frac{2}{7}(0.2)^j+\frac{5}{7}(-0.5)^j\right)L^j$$

We need to find the long run value, $\psi(1)=\sum_j\psi_j$, which we get from (13), $\psi(L)\varphi(L)=1$, which, translating to usual polynomials and power series, must be equivalent to $\psi(z)\varphi(z)=1$. Since we are dealing with usual functions here, this implies that $\psi(z)=\dfrac{1}{\varphi(z)}$ (valid for all z such that $|z|\le 1$ as long as the roots of $\varphi(z)$ are outside the unit circle).

In particular: $\psi(1)=\dfrac{1}{\varphi(1)}=\dfrac{1}{1+0.3-0.1}=\dfrac{1}{1.2}=\dfrac{6}{5}$

As just after (13) we get

$$\mu=E(Y_t)=E\left(\psi(1)\varphi_0+\psi(L)\varepsilon_t\right)=\psi(1)\varphi_0=\frac{6}{5}\cdot 1=\frac{6}{5}$$

(End of example)

Notice the last argument of the example. Due to theorem 2 all coefficients in the product $\psi(L)\varphi(L)$ are the same as in the product $\psi(z)\varphi(z)$ between usual functions. Since the equation $\psi(z)\varphi(z)=1$ has the solution, $\psi(z)=\frac{1}{\varphi(z)}$, it is natural to define a new operator, $\frac{1}{\varphi(L)}$, which, by definition, simply means the filter $\psi(L)$. For example, using this new operator on a constant c, we get

$$\frac{1}{\varphi(L)}c \overset{\text{def}}{=} \psi(L)c \overset{\text{Th. 2}}{=} \left(\psi(z)c\right)_{z=1} = \left(\frac{1}{\varphi(z)}c\right)_{z=1} = \frac{c}{\varphi(1)}$$

With this (extremely much used) trick of introducing $\frac{1}{\varphi(L)}$, we obtain an elegant expression (not to say algebra) of solutions, e.g., of an ARMA(p,q) as above

$$\varphi(L)Y_t = w_t = \varphi_0 + \theta(L)\varepsilon_t \Leftrightarrow$$

(21) $$Y_t = \frac{1}{\varphi(L)}\left(\varphi_0 + \theta(L)\varepsilon_t\right) \overset{\text{Th. 2}}{=} \frac{1}{\varphi(L)}\varphi_0 + \frac{1}{\varphi(L)}\theta(L)\varepsilon_t = \frac{\varphi_0}{\varphi(1)} + \frac{\theta(L)}{\varphi(L)}\varepsilon_t$$

so the solution in this case is to express $\frac{\theta(L)}{\varphi(L)}$ as a linear filter,

$\delta(L) = 1 + \delta_1 L + \delta_2 L^2 + \delta_3 L^3 + \cdots$. i.e., solving

(22) $$\varphi(L)\delta(L) = \theta(L)$$

which leads to same homogeneous difference equation as for $\psi(L)$, but with different initial conditions (see summary below).

In the AR(p) case:

(22) $$Y_t = \frac{1}{\varphi(L)}\left(\varphi_0 + \varepsilon_t\right) = \frac{\varphi_0}{\varphi(1)} + \frac{1}{\varphi(L)}\varepsilon_t, \text{ where } \frac{1}{\varphi(L)} = \psi(L)$$

For example, in the AR(1) case that we solved last time,

$$Y_t = \varphi_0 + \varphi_1 Y_{t-1} + \varepsilon_t \quad \text{or}$$

$$Y_t - \varphi_1 Y_{t-1} = \varphi_0 + \varepsilon_t \quad \Rightarrow \varphi(L) = 1 - \varphi_1 L \quad \Rightarrow \varphi(z) = 1 - \varphi_1 z$$

which has only one root, $z = 1/\varphi_1$, that must be outside the unit circle to generate a causal filter, i.e., $|\varphi_1| < 1$.

Since then $\frac{1}{\varphi(z)} = \frac{1}{1-\varphi_1 z} = \sum_{j=0}^{\infty} (\varphi_1 z)^j = \sum_{j=0}^{\infty} \varphi_1^j z^j \Rightarrow \frac{1}{\varphi(L)} = \sum_{j=0}^{\infty} \varphi_1^j L^j$

we get the solution

$$Y_t = \frac{\varphi_0}{\varphi(1)} + \frac{1}{\varphi(L)}\varepsilon_t = \frac{\varphi_0}{1-\varphi_1} + \varepsilon_t + \varphi_1 \varepsilon_{t-1} + \varphi_1^2 \varepsilon_{t-2} + \cdots$$

as we found before.

For completeness sake a summary of the general causal filter solution is given

Summary of the causal filter solution for general *p*.

We want a causal solution of
$Y_t - \varphi_1 Y_{t-1} - \cdots - \varphi_p Y_{t-p} = w_t$, or, with corresponding lag polynomial,
(21) $\varphi(L) Y_t = w_t$, where $\varphi(L) = 1 - \varphi_1 L - \varphi_2 L^2 - \cdots - \varphi_p L^p$ is the lag polynomial and $\varphi(\mathrm{z}) = 1 - \varphi_1 z - \varphi_2 z^2 - \cdots - \varphi_p z^p$ the companion polynomial.

The reversed companion polynomial is

(22) $$h(x) = x^p - \varphi_1 x^{p-1} - \varphi_2 x^{p-2} - \cdots - \varphi_p$$

that, according to the fundamental theorem of algebra (see appendix) always must have *p* roots, $r_1, r_2, \ldots, r_p$, some of which may be equal, and some of which may be complex. The number of times a root occurs is called the **multiplicity** of the root.

> [For example, the polynomial, $(x-2)^3(x+0.5)^2(x-0.8)$ has order 6, but only 3 different roots, 2, -0.5, and 0.8, with multiplicities 3, 2, and 1 respectively.]

So, in general, (22) has *s* different roots, $r_1, r_2, \ldots, r_s$, with multiplicities, $m_1, m_2, \ldots, m_s$, respectively, where $m_1 + m_2 + \ldots + m_s = p$.

AR(*p*) case: The filter solution, $\psi(L) = \sum_{j=0}^{\infty} \psi_j L^j$, in the must satisfy (13), i.e.,

(23) $$\psi(L)\varphi(L) = 1$$

which, as above for $p = 2$, leads to the homogeneous difference equation for $\{\psi_t\}$ for $t \geq p$,

(24) $$\psi_t - \varphi_1 \psi_{t-1} - \cdots - \varphi_p \psi_{t-p} = 0, \quad \text{for } t \geq p, \quad \text{or} \quad \varphi(L)\psi_t = 0$$

The *p* initial conditions for AR(*p*) are

(25) $$\begin{aligned} &\psi_0 = 1, \quad \psi_1 = \varphi_1, \quad \psi_2 = \varphi_1 \psi_1 + \varphi_2, \quad \psi_3 = \varphi_1 \psi_2 + \varphi_2 \psi_1 + \varphi_3, \ldots \\ &\ldots, \psi_{p-1} = \varphi_1 \psi_{p-2} + \varphi_2 \psi_{p-3} + \cdots + \varphi_{p-1} \end{aligned}$$

The ARMA(*p*,*q*) case: The solution is

$$Y_t = \frac{\varphi_0}{\varphi(1)} + \frac{\theta(L)}{\varphi(L)}\varepsilon_t = \frac{\varphi_0}{\varphi(1)} + \delta(L)\varepsilon_t, \text{ where } \delta(L) = \frac{\theta(L)}{\varphi(L)} \text{ satisfies}$$

(26) $\varphi(L)\delta(L) = \theta(L)$

which leads to the same homogeneous difference equation as for $\{\psi_t\}$,

(27) $\delta_t - \varphi_1\delta_{t-1} - \cdots - \varphi_p\delta_{t-p} = 0,$ for $t \geq \max(p, q+1)$

with initial conditions for $t < \max(p, q+1)$

(28)
$$\begin{aligned} \delta_0 &= 1 \\ \delta_1 - \varphi_1\delta_0 &= \theta_1 \\ \delta_2 - \varphi_1\delta_1 - \varphi_2\delta_0 &= \theta_2 \\ \delta_3 - \varphi_1\delta_2 - \varphi_2\delta_1 - \varphi_3\delta_0 &= \theta_3 \\ &\vdots \end{aligned}$$

The general solution of (24) and (27) is

(**i**) If all roots, $r_1, r_2, \ldots, r_p$, of the reversed companion polynomial are different, then

$$\psi_t (\text{or } \delta_t) = c_1 r_1^t + c_2 r_2^t + \cdots + c_p r_p^t$$

where the constants, $c_1, c_2, \ldots, c_p$ are determined such that the initial conditions (25) are fulfilled.

(**ii**) If only *s* roots, $r_1, r_2, \ldots, r_s$, are different with multiplicities, $m_1, m_2, \ldots, m_s$, respectively, the general solution is

$$\psi_t (\text{or } \delta_t) = p_1(t) r_1^t + p_2(t) r_2^t + \cdots + p_s(t) r_s^t$$

where $p_i(t)$ is a $(m_i - 1)$ -degree polynomial in *t*,

$p_i(t) = c_{i0} + c_{i1}t + c_{i2}t^2 + \cdots + c_{i(m_i-1)}t^{m_i-1}$, where the coefficients are determined such that the initial conditions (25) (or (28)) are fulfilled.

(**iii**) The solution filter is causal stationary if and only if all roots of the reversed companion polynomial are inside the unit circle.

Appendix. Crash course I in complex numbers – The unit circle.

We introduce the symbol $i=\sqrt{-1}$ which is interpreted as a "number" that satisfies $i^2=-1$, and that we call "imaginary" or "complex". Then the equation

(1) $4+(x-1)^2=x^2-2x+5=0$

gets two (complex) roots (notice that the expression on the left is larger than zero always)[1]. The roots, r_1, r_2, are found in the usual way:

$$x=\frac{1}{2}\left(2\pm\sqrt{4-4\cdot 5}\right)=1\pm\frac{1}{2}\sqrt{-16}=1\pm\frac{1}{2}\sqrt{-1}\sqrt{16}=1\pm 2i$$

The polynomial in (1) can then be factorized (as in the real case)

$$x^2-2x+5=(x-r_1)(x-r_2)=(x-1-2i)(x-1+2i)$$

which can be checked by multiplying out the right side (assuming that all common algebraic operations are valid):

$$(x-1-2i)(x-1+2i)=(x-1)^2+(x-1)2i-2i(x-1)-4i^2=(x-1)^2-4(-1)=$$
$$=(x-1)^2+4$$

Properties

1) All complex numbers can be written uniquely as $z=a+ib$ where *a* and *b* are real. *z* can be interpreted geometrically as a point, (a,b), in the usual two-dimensional plane, $z=a+ib=(a,b)$. Hence *i* itself is represented by the point (0,1) which corresponds to the unit vector on the y-axis.

[1] This was the way complex numbers were introduced originally. People noticed that by doing this, they achieved that every *n*-dimensional polynomial always has *n* roots. This they considered a great advantage. It took some time, however, before one managed to prove that this trick could not lead to contradictions in math.

2) Real numbers are now interpreted as special complex numbers, i.e., all points on the x-axis, where $b=0$, i.e., $x=a+i(0)=(a,0)$.
3) It can be shown that all usual algebra for real numbers can be used for complex numbers (e.g., $1/i=i/(ii)=i/(-1)=-i=(0,-1)$).
4) The *absolute value* of $z=a+bi$, also called *the modulus* (written $|z|$), is defined as the distance to the origin of the point (a,b), $\sqrt{a^2+b^2}$, (from Pytagoras' theorem). The modulus of a real number $(a,0)$ then becomes $\sqrt{a^2+0^2}=\sqrt{a^2}=|a|$, i.e., the usual absolute value of a.
5) The modulus satisfies $|z_1z_2|=|z_1|\cdot|z_2|$.

Proof: Let $z_j=a_j+ib_j$. Then

$z_1z_2=(a_1+ib_1)(a_2+ib_2)=(a_1b_1-a_2b_2)+i(a_1b_2+a_2b_1)$, from which

$$|z_1z_2|=\sqrt{a_1^2a_2^2+b_1^2b_2^2-2a_1a_2b_1b_2+a_1^2b_2^2+a_2^2b_1^2+2a_1b_2a_2b_1}=$$
$$=\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}=|z_1|\cdot|z_2|$$

In particular we get $|z^2|=|zz|=|z|\cdot|z|=|z|^2$, and more generally, $|z^k|=|z|^k$.

6) If $|z|<1$, we must have that $z^k\to 0$ when $k\to\infty$.

Proof: The distance to the origin for z^k is given by $|z^k|=|z|^k\to 0$, since $|z|$ is a real positive number <1.

Example: Let $z=(1+i)/2$, from which $|z|=\sqrt{1/4+1/4}=1/\sqrt{2}<1$. We find

$$z^2=\left(\frac{1}{2}+\frac{1}{2}i\right)^2=\frac{1}{4}+2\frac{1}{2}\cdot\frac{1}{2}i+\left(\frac{1}{2}i\right)^2=\frac{1}{4}+\frac{i}{2}-\frac{1}{4}=\frac{i}{2}$$
$$z^3=z^2\cdot z=\frac{i}{2}\left(\frac{1}{2}+\frac{i}{2}\right)=\frac{i}{4}+\frac{i^2}{4}=-\frac{1}{4}+\frac{i}{4}$$
$$z^4=\left(-\frac{1}{4}+\frac{i}{4}\right)\left(\frac{1}{2}+\frac{i}{2}\right)=\left(-\frac{1}{8}-\frac{1}{8}\right)+i\left(-\frac{1}{8}+\frac{1}{8}\right)=-\frac{1}{4}$$

… etc. (Locate these points in the plane. You will see that z^k moves toward 0 along a spiral in the plane).

7) **Complex conjugates.** If $z=a+ib$ is a complex number, we define the complex conjugate of z (written $\bar{z}$) as $\bar{z}=a-ib$. We have

i. $\overline{z_1+z_2}=\bar{z}_1+\bar{z}_2$

[**Proof.**

$z_j=a_j+ib_j \Rightarrow z_1+z_2=(a_1+a_2)+i(b_1+b_2)$

$\Rightarrow \overline{z_1+z_2}=(a_1+a_2)-i(b_1+b_2)=a_1-ib_1+a_2-ib_2$

ii. $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ implying $\overline{z^k} = \bar{z}^k$

[Proof.

$\overline{z_1 z_2} = \overline{(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)}$

$= (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + a_2 b_1)$

$\bar{z}_1 \, \bar{z}_2 = (a_1 - ib_1)(a_2 - ib_2)$

$= (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + a_2 b_1)$

iii. $z\bar{z} = |z|^2$ [$(a+ib)(a-ib) = a^2 + b^2$]

iv. z real $\Rightarrow \bar{z} = z$ [$\bar{z} = \overline{a + i0} = a - i0 = a = z$]

The unit circle

is defined as all complex numbers, *z*, that has a constant distance = 1 to the origin, i.e., such that $|z| = |a+bi| = \sqrt{a^2+b^2} = 1$. Such points in the plane form a circle around the origin with constant radius equal to 1. *z* is inside the circle if $|z| < 1$ since then the distance to the origin is smaller than 1, and *z* lies outside the circle if $|z| > 1$.

The fundamental theorem of algebra

says that any *p*-degree polynomial, $a_0 + a_1 z + a_2 z^2 + \cdots + a_p z^p$, has exactly *p* roots, $r_1, r_2, \ldots, r_p$, some of which may be equal but are counted as many times they occur.

An important consequence of 7) is that if the polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_p z^p$ has real coefficients (a_j) only, and if *r* is a complex root (i.e., such that $p(r) = 0$, then the conjugate, $\bar{r}$, is also a root. This follows since

$$0 = \bar{0} = \overline{p(r)} = \overline{a_0 + a_1 r + a_2 r^2 + \cdots + a_p r^p} \overset{7)}{=}$$

$$\bar{a}_0 + \bar{a}_1 \bar{r} + \bar{a}_2 \bar{r}^2 + \cdots + \bar{a}_p \bar{r}^p = a_0 + a_1 \bar{r} + a_2 \bar{r}^2 + \cdots + a_p \bar{r}^p$$

Hence, if complex roots occur in real polynomials, they always occur in pairs, as r and $\bar{r}$.

Examples.

From the fundamental theorem follows that the equation $x^k = 1$, for any *k*, has *k* solutions always.

For example, $x^3 - 1$ is a third degree polynomial and must have three roots. Using *polar coordinates* (that we will not take up here although very important tool) as described in Ragnar's lecture notes on complex numbers on the web page for the 2011 course, the three exact roots are

$r_1 = 1$, $\quad \mathrm{r}_2 = \cos(2\pi/3) + i\sin(2\pi/3) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and

$\mathrm{r}_3 = \cos(4\pi/3) + i\sin(4\pi/3) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

(a slight mistake in Ragnar's example)

Stata. In practice it is, maybe, better to use a computer to find roots. For example, the routine Mata in Stata is easy to use (the command `polyroots` calculates roots of polynomials):

The command, `mata,` brings you into Mata shown by the colon prompt " : " instead of the usual point " . ". To get out of Mata you just write the command `end` and enter. Mata identifies a polynomial by a vector of coefficients with the lowest degree coefficients first. For example the polynomial, $x^2 - 2x + 5$, is described by the vector $(5, -2, 1)$, and the polynomial, $x^3 - 1$, by $(-1, 0, 0, 1)$.

```
. mata
------------------------------------------------- mata (type end to exit) -
: polyroots((5,-2,1))
            1         2
    +-------------------+
  1 |  1 - 2i    1 + 2i  |                        (As we got in the beginning)
    +-------------------+

: polyroots((-1,0,0,1))                          (The roots of x^3-1)
                       1                    2                   3
    +---------------------------------------------------------------+
  1 |   -.5 - .866025404i    -.5 + .866025404i                  1   |
    +---------------------------------------------------------------+

: r=polyroots((-1,0,0,1))            (I want access to the first root r[1])
```

```
: r
                      1                   2                   3
    +-------------------------------------------------------------+
  1 |  -.5 - .866025404i   -.5 + .866025404i                  1  |
    +-------------------------------------------------------------+

: r[1]
  -.5 - .866025404i

: r[1]^3                           (Checking that r[1] really is a root)
  1

: end
--------------------------------------------------------------------------
```

.

(Further details on complex number can be read in Ragnar's lecture notes on this on the web page for the 2011 course.)