HG
Jan 2014 ECON 5101

Exercises II - 10 Feb 2014 (with answers)

Exercise 1.
Read section 8 in lecture notes 3 (LN3) on the common factor problem in ARMA-processes.

Consider the following process

(1) $$Y_t = 0.4Y_{t-1} + 0.45Y_{t-2} + \varepsilon_t + \varepsilon_{t-1} + 0.25\varepsilon_{t-2}$$

where $\varepsilon_t \sim WN(0,\sigma^2)$.

This is an ARMA process of the form $\varphi(L)Y_t = \theta(L)\varepsilon_t$ without constant term.

A. Investigate if the two lag-polynomials have a common factor. If so, reformulate the difference equation specification for $\{Y_t\}$ to a proper ARMA specification.

B. Is there a causal stationary solution for $\{Y_t\}$? If yes, write up the solution as a $MA(\infty)$ process. Is the MA specification invertible? If so, write up the $AR(\infty)$ solution for ε_t .

ANSWER:
A: The companion polynomials:

	Roots	
$\varphi(z) = 1 - 0.4z - 0.45z^2$	-2	$10/9 = 1.1111...$
$\theta(z) = 1 + z + 0.25z^2$	-2	-2

Factorized:

$$\varphi(z) = (1+0.5z)(1-0.9z)$$
$$\theta(z) = (1+0.5z)^2$$

To have a proper ARMA we should cancel the common factor, giving new lag polynomials

$$\varphi(L) = 1 - 0.9L$$
$$\theta(L) = 1 + 0.5L$$

The proper ARMA(1,1) specification

(1.1) $$Y_t - 0.9Y_{t-1} = \varepsilon_t + 0.5\varepsilon_{t-1}$$

B: The solution of (1.1) is causal stationary since $|\varphi| = 0.9 < 1$.

$$Y_t = \frac{1 + 0.5L}{1 - 0.9L}\varepsilon_t$$

We need to find the $\psi\text{'s}$ in

$$\frac{1 + 0.5z}{1 - 0.9z} = 1 + \psi_1 z + \psi_2 z^2 + \cdots \quad \text{or}$$

(1.2) $$1 + 0.5z = (1 - 0.9z)\left(1 + \psi_1 z + \psi_2 z^2 + \cdots\right)$$

Multiplying out the right side (putting $\varphi = 0.9, \quad \theta = 0.5$)

$$\begin{aligned} 1 + \theta z &= 1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \cdots \\ &\qquad - \varphi z - \varphi\psi_1 z^2 - \varphi\psi_2 z^3 - \cdots = \\ &= 1 + (\psi_1 - \varphi)z + (\psi_2 - \varphi\psi_1)z^2 + (\psi_3 - \varphi\psi_2)z^3 + \cdots \end{aligned}$$

Hence (using the uniqueness of power series coefficients (see appendix to the exercises II))

$$\begin{aligned} (\psi_1 - \varphi) = \theta \quad &\Rightarrow \quad \psi_1 = \varphi + \theta \\ \psi_j = \varphi\psi_{j-1} \quad &\Rightarrow \quad \psi_j = \varphi^{j-1}\psi_1 = \\ &= \varphi^{j-1}(\varphi + \theta) = (0.9)^{j-1}(1.4) \quad \text{for } j = 2, 3, \ldots \end{aligned}$$

The process is invertible since $|\theta| = 0.5 < 1$. The AR(∞) solution is

$$\varepsilon_t = \frac{1-0.9L}{1+0.5L} Y_t = (1 + \delta_1 L + \delta_2 L^2 + \cdots) Y_t$$

Putting $\varphi = 0.9, \quad \theta = 0.5$, and using the companion forms, we get

$$\begin{aligned}
1 - \varphi z &= (1 + \theta z)(1 + \delta_1 z + \delta_2 z^2 + \cdots) = \\
&= 1 + \delta_1 z + \delta_2 z^2 + \delta_3 z^3 + \cdots \\
&\quad + \theta z + \theta\delta_1 z^2 + \theta\delta_2 z^3 + \cdots = \\
&= 1 + (\delta_1 + \theta) z + (\delta_2 + \theta\delta_1) z^2 + (\delta_3 + \theta\delta_2) z^3 + \cdots
\end{aligned}$$

Hence

$$\begin{aligned}
&\delta_1 + \theta = -\varphi \quad \Rightarrow \quad \delta_1 = -(\varphi + \theta) = -1.4 \\
&\delta_j = (-\theta)^{j-1} \delta_1 = -(-\theta)^{j-1}(\varphi + \theta) = -(-0.5)^{j-1}(1.4) \text{ for } j = 2, 3, \ldots
\end{aligned}$$

Some values:

j	0	1	2	3	4	5	10	20	40
ψ_j	1	1.4	1.26	1.134	1.021	0.919	0.542	0.189	0.023
δ_j	1	-1.4	0.70	-0.350	0.175	-0.0875	0.003	2.67E-06	2.55E-12

Exercise 2.

A. Suppose that $Y_t \sim ARMA(p,q)$ and causal, satisfying) $\varphi(L)Y_t = \varphi_0 + \theta(L)\varepsilon_t$, where $\varphi(L) = 1 - \varphi_1 L - \cdots - \varphi_p L^p, \quad \theta(L) = (1 + \theta_1 L + \cdots + \theta_q L^q)$. Introduce the centered series, $y_t = Y_t - \mu$, leading to $\varphi(L) y_t = \theta(L)\varepsilon_t$ without the constant φ_0, and where $E(y_t) = 0$ (see exercise 3 of seminar1).

Explain why statement (7) on page 5 of LN3, saying $\varphi(L)\gamma(h) = 0$ and $\varphi(L)\rho(h) = 0$ for $h \geq \max(p, q+1)$, is true, where $\gamma(h), \rho(h)$ are the autocovariance function and acf respectively.

[**Hint.** Assume $h \geq \max(p, q+1)$ and write

$$\gamma(h)=E(y_{t+h}y_t)=E\left(\varphi_1 y_{t+h-1}y_t+\cdots+\varphi_p y_{t+h-p}y_t+\varepsilon_{t+h}y_t+\cdots+\theta_q\varepsilon_{t+h-q}y_t\right)$$

etc.]

ANSWER: Continuing the hint we get

(*) $\gamma(h)=E(y_{t+h}y_t)=\varphi_1\gamma(h-1)+\cdots+\varphi_p\gamma(h-p)+E(\varepsilon_{t+h}y_t)+\cdots+\theta_q E(\varepsilon_{t+h-q}y_t)$

(i) If $h<p$, the "answer" I gave on the seminar was not good. The point is rather: Suppose, e.g., $h=p-1$. Then, remembering that $\gamma(-1)=\gamma(1)$, (*) becomes

$$\begin{aligned}\gamma(h)=E(y_{t+h}y_t)&=\varphi_1\gamma(h-1)+\cdots+\varphi_{p-2}\gamma(1)+\varphi_{p-1}\gamma(0)+\varphi_p\gamma(-1)+E(\varepsilon_{t+h}y_t)+\cdots=\\&=\varphi_1\gamma(h-1)+\cdots+\varphi_{p-2}\gamma(1)+\varphi_{p-1}\gamma(0)+\varphi_p\gamma(-1)+E(\varepsilon_{t+h}y_t)+\cdots=\\&=\varphi_1\gamma(h-1)+\cdots+(\varphi_{p-2}+\varphi_p)\gamma(1)+\varphi_{p-1}\gamma(0)+E(\varepsilon_{t+h}y_t)+\cdots\end{aligned}$$

and we see that the difference equation part for $\gamma(h)$ has changed (the coefficients are no longer the same). . Therefore, we must have $h\ge p$, for the difference equation for $\gamma(h)$ to hold.

(ii) Since the (causal) solution for y_t depends on $\varepsilon_t,\varepsilon_{t-1},\varepsilon_{t-2},\ldots$, we must have $h-q\ge 1$ (or $h\ge q+1$) in order that all the last ε -terms become zero.

Hence $h\ge\max(p,q+1)$.

B. **Introduction**. If $Y_t\sim AR(1)$ with $y_t=\varphi y_{t-1}+\varepsilon_t$ and $|\varphi|<1$, we get from page 3 in LN2 that the autocovariance function, $\gamma_1(-h)=\gamma_1(h)=\sigma^2\dfrac{\varphi^h}{1-\varphi^2}$ which implies that the acf is $\rho(-h)=\rho(h)=\varphi^h$ for $h=0,1,2,\ldots$

We are interested in this exercise to find out what the effect is on the AR(1) acf, $\rho(h)$ by adding a MA-term, $\theta\varepsilon_{t-1}$ to $y_t=\varphi y_{t-1}+\varepsilon_t$.

Assume therefore that $y_t-\varphi y_{t-1}=\varepsilon_t+\theta\varepsilon_{t-1}$, i.e., an ARMA(1,1), where $|\varphi|<1,\quad |\theta|<1,$ and $\varepsilon_t\sim WN(0,\sigma^2)$. We need the autocovariance, $\gamma(h)$ and acf $\rho(h)$ for y_t in this new situation. First $\gamma(h)$: Using (8) on page 5 in LN3 and **A.**, we have

(i) $\gamma(h)-\varphi\gamma(h-1)=0$ for $h=2,3,\cdots$

with initial conditions

(ii) $\gamma(0)-\varphi\gamma(-1)=\sigma^2(\delta_0+\theta\delta_1)$, i.e., $\gamma(0)-\varphi\gamma(1)=\sigma^2(\delta_0+\theta\delta_1)$

(iii) $\gamma(1)-\varphi\gamma(0)=\sigma^2(\theta\delta_0)$

where δ_0,δ_1 are found from $\dfrac{1+\theta z}{1-\varphi z}=\delta_0+\delta_1 z+\delta_2 z^2+\cdots$

Question. Show that $\delta_0=1$ and $\delta_1=\varphi+\theta$

[**Hint.** Write the lag-equation (in terms of companion series) $1+\theta z=(1-\varphi z)(\delta_0+\delta_1 z+\delta_2 z^2+\cdots)$. Then multiply out the right side sufficiently to determine δ_0,δ_1 .]

ANSWER: This was done in exercise 1B

C. Show first from ii. and iii. that

$$\gamma(0)=\sigma^2\frac{1+2\theta\varphi+\theta^2}{1-\varphi^2}$$

$$\gamma(1)=\sigma^2\frac{(\theta+\varphi)(1+\theta\varphi)}{1-\varphi^2}$$

and then that $\gamma(h)=\sigma^2\dfrac{(\theta+\varphi)(1+\theta\varphi)}{1-\varphi^2}\varphi^{h-1}$ for $h\ge 1$.

ANSWER: $\delta_0=1,\quad \delta_1=\theta+\varphi,\quad \gamma(h)=\varphi\gamma(h-1)$ for $h=2,3,\ldots$ gives

$\gamma(h)=\varphi^{h-1}\gamma(1)$ for $h=2,3,\ldots$

From ii., iii. we get

(*) $\gamma(0) = \sigma^2(1+\theta(\theta+\varphi)) + \varphi\gamma(1)$

$\gamma(1) = \theta\sigma^2 + \varphi\gamma(0) = \theta\sigma^2 + \varphi(1+\theta(\theta+\varphi))\sigma^2 + \varphi^2\gamma(1)$

$\Rightarrow$

$(1-\varphi^2)\gamma(1) = \sigma^2\left[\theta+\varphi+\theta\varphi(\theta+\varphi)\right] = \sigma^2(\theta+\varphi)(1+\theta\varphi)$

$\Rightarrow$

$$\gamma(1) = \sigma^2 \frac{(\theta+\varphi)(1+\theta\varphi)}{1-\varphi^2}$$

Substituting in (*) gives

$$\gamma(0) = \sigma^2\left[1+\theta(\theta+\varphi)+\varphi\frac{(\theta+\varphi)(1+\theta\varphi)}{1-\varphi^2}\right] =$$

$$= \sigma^2\frac{1-\varphi^2+\theta(\theta+\varphi)(1-\varphi^2)+\varphi(\theta+\varphi)(1+\varphi\theta)}{1-\varphi^2} =$$

$$= \sigma^2\frac{1-\varphi^2+(\theta+\varphi)\left[\theta(1-\varphi^2)+\varphi(1+\varphi\theta)\right]}{1-\varphi^2} =$$

$$= \sigma^2\frac{1-\varphi^2+(\theta+\varphi)^2}{1-\varphi^2} = \sigma^2\frac{1-\varphi^2+\varphi^2+2\varphi\theta+\theta^2}{1-\varphi^2}$$

Hence

$$\gamma(0) = \sigma^2\frac{1+2\varphi\theta+\theta^2}{1-\varphi^2}, \quad \gamma(1) = \sigma^2\frac{(\theta+\varphi)(1+\theta\varphi)}{1-\varphi^2}, \quad \text{and}$$

(**)

$$\gamma(h) = \sigma^2\frac{(\theta+\varphi)(1+\theta\varphi)}{1-\varphi^2}\varphi^{h-1} \quad \text{for } h = 2,3,\ldots$$

D. Show that the acf of y_t can be expressed as $\rho(h) = \dfrac{(\theta+\varphi)(1+\theta\varphi)}{\varphi(1+2\theta\varphi+\theta^2)}\rho_1(h)$,

where $\rho_1(h)$ is the acf of the AR(1) process.

ANSWER: From (**) we get

$$\rho(h) = \frac{(\theta+\varphi)(1+\theta\varphi)}{(1+2\theta\varphi+\theta^2)}\varphi^{h-1} \quad \text{for } h = 1, 2, 3, \ldots$$

Considering, $\rho_1(h) = \varphi^h$, we get $\rho(h) = \dfrac{(\theta+\varphi)(1+\theta\varphi)}{\varphi(1+2\theta\varphi+\theta^2)}\rho_1(h)$.

E. The constant factor in front of $\rho_1(h)$, characterize the effect θ has on the AR(1) acf, $\rho_1(h)$. Look at this effect (factor) in the two special cases, $\varphi = 0.9$ and $\varphi = -0.2$, i.e., for which values of θ will the AR(1) acf increase and for which values will it decrease?

[Hint. The factor in front of $\rho_1(h)$ is not so easy to discuss analytically. The best way, in my view, is simply to plot the factor as a function of θ for $-1 < \theta < 1$ in the two cases. The easiest is probably to plot it with a computer. This is, for example, easy in Stata. The following command, for example,

```
twoway function y=2*x^2-1, range(-1 2.5)
```

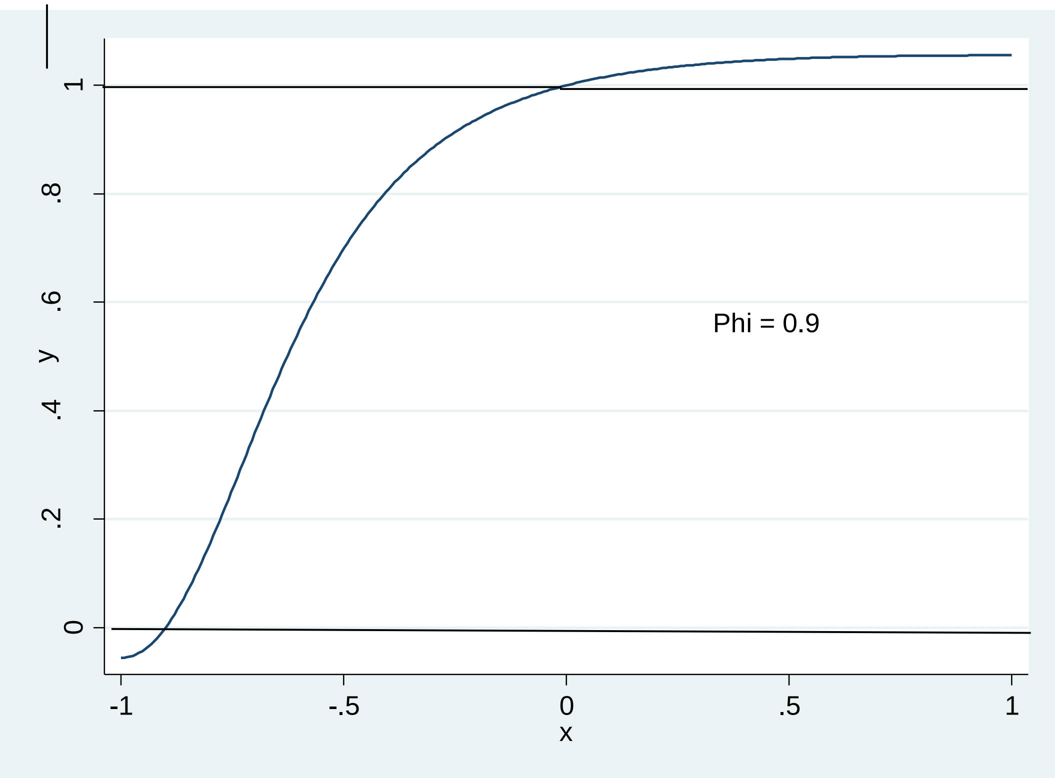
plots the function $y = 2x^2 - 1$ for $-1 < x < 2.5$]

ANSWER: The factor in front of $\rho_1(h)$ represents the effect the MA-term has on the acf of AR(1). Writing *x* for θ, the factor is the function

$$g(x) = \frac{(x+\varphi)(1+\varphi x)}{\varphi(1+2\varphi x+x^2)}$$

Case $\varphi = 0.9$: Stata-plot :

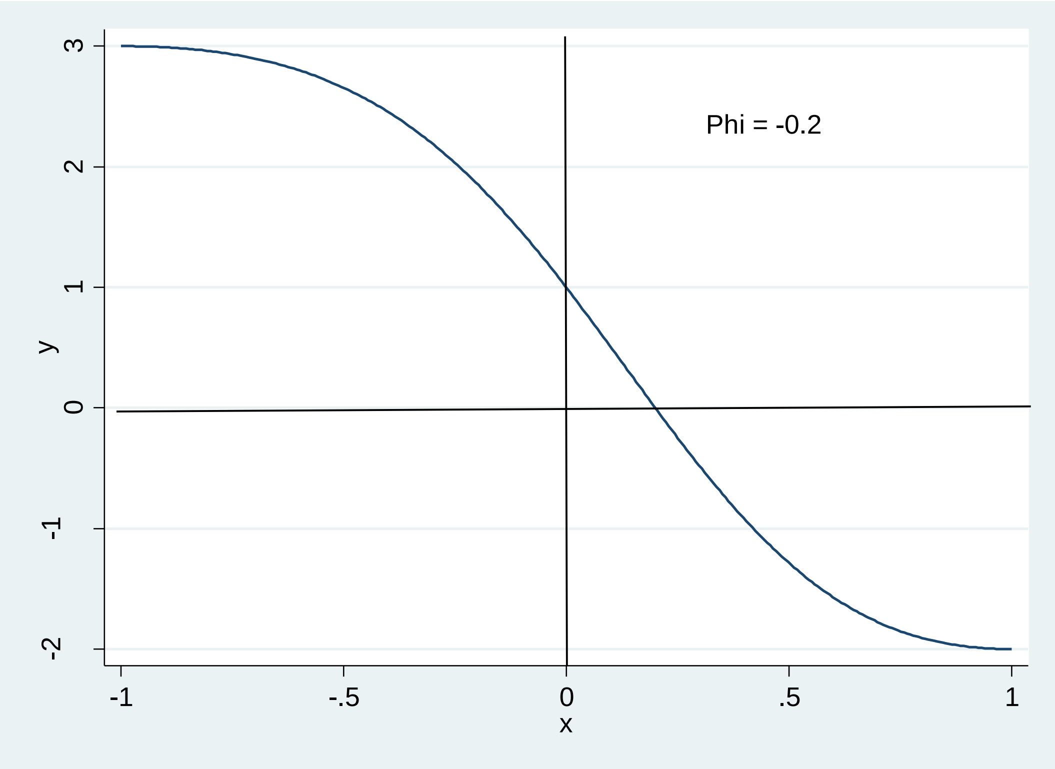
```
twoway function y=(x+.9)*(1+.9*x)/(.9*(1+2*.9*x+x^2)), range(-1 1)
```



$$g_{\max} = g(1) = (1+\varphi)/2\varphi = 1.056, \quad g_{\min} = g(-1) = -(1-\varphi)/2\varphi = -0.056$$

(We see that if θ increases from 0 (e.g.), $\rho(h)$ will increase also from $\rho_1(h)$, but not much.)

Case $\varphi = -0.2$:



F. For any φ with $|\varphi| < 1$, is there any value of θ that turns $\{y_t\}$ into white noise? If so, for which value?

ANSWER: From the figures we see that there is a θ that makes $\rho(h) = 0$ for $h \neq 0$. From the formula we see that this happens for $\theta = -\varphi$. We also see from the solution series

$$y_t = \frac{1+\theta L}{1-\varphi L}\varepsilon_t$$

that there is a common factor when $\theta = -\varphi$, so that $\{y_t\}$ is white noise for this value of θ.

(For the **appendix** on power series see the original exercise text)