Complex numbers.
Reference note to lecture 9 ECON 5101 Time Series Econometrics

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1 Introduction

This note gives the definitions and theorems related to complex numbers that are used in the course. The note is a "stand alone" supplement to Hamilton's book and there has been no attempt to synchronize the notation.

2 Definition and representations

z is a complex number if

$$z = a + ib$$

where a,b are real numbers and $i = \sqrt{-1}$ is an imaginary "number" that satisfies $i^2 = -1$. We often write

$$Re(z) = a$$

for the *real part* of z and

$$Im(z) = b$$

for the *imaginary part* of z.

A real number like a can often be interpreted as a special complex number by writing it as $a + i \cdot 0$. Similarly: A way of writing i is $0 + i \cdot 1$.

Definition 1 (Addition and multiplication) *We have the following definitions for addition and multiplication of complex numbers:*

$$\begin{aligned} z_1 + z_2 &= (a_1 + ib_1) + (a_2 + ib_2) \overset{def}{=} (a_1 + a_2) + i(b_1 + b_2) \\ z_1 z_2 &= (a_1 + ib_1)(a_2 + ib_2) \overset{def}{=} (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_1 b_2) \end{aligned}$$

With the use of these definitions we have all the rules we need for working with complex numbers (except for inequalities).

A complex number $(a + ib)$ is often shown graphically in a diagram with the real part (a) along the horizontal axis, and the imaginary part (b) along the vertical axis. Note that "i" is the unit of measurement on the vertical axis.

Definition 2 (Norm) *The distance from $z = a + ib$ to origo is the* norm *of z:*

$$|z| = \sqrt{a^2 + b^2}.$$

Norm and modulus are synonyms.

Definition 3 (Conjugate) *The complex conjugate to $z = a + ib$ is*

$$\overline{z} = a - ib$$

With these definitions we have for example: $\overline{i+1} = 1 - i$, $\overline{i} = -i$ and $\overline{2} = 2$.

Definition 4 (Trigonometric form) *The trigonometric representation of the complex number z is:*

$$z = a + ib = |z| \cos\theta + i\,|z| \sin\theta = |z|\,(\cos\theta + i \sin\theta)$$

where θ, called the argument *is given by*

$$\sin\theta = \frac{b}{\sqrt{a^2 + b^2}} = \frac{b}{|z|} \tag{2.1}$$

$$\cos\theta = \frac{a}{\sqrt{a^2 + b^2}} = \frac{a}{|z|} \tag{2.2}$$

The representation is often referred to as the polar-coordinate form.

Note that the argument θ is not uniquely defined: If θ fits in (2.1) and (2.2), then $\theta + 2k\pi$, $k = \pm 1, \pm 2, \ldots$ also fit. A convention is to choose θ to be in region $-\pi < \theta \leq \pi$. If $-\pi/2 < \theta < \pi/2$ we find θ as

$$\theta = \arctan(\frac{b}{a})$$

since $\tan(\theta) = \sin(\theta)/\cos(\theta) = b/a$.

3 Frequency and period

In time series analysis, it is practical to measure the *frequency* in *cycles per unit of time*, which we denote by v. If the unit of time is 1 year, the frequencies could be for example 1/8, 1/4, 1 or maybe 2 cycles per year. Hence in this application the relevant argument in the trigonometric function is positive. In classical harmonic analysis, the frequency is measured in terms of radians *per unit of time*, and we have the equation

$$\theta = 2\pi v$$

relating the frequency in cycles per unit of time to radians per unit of time. From the properties of the cosine function we have for example: $v = 0 \Rightarrow \cos(2\pi 0) = 1$, $v = 1/4 \Rightarrow \cos(\pi/2) = 0$, $v = 3/4 \Rightarrow \cos(3\pi/2) = 0$, $v = 1 \Rightarrow \cos(2\pi) = 1$.

In line with this convention, v defined as *cycles per unit of time*, we therefore determine the argument of the trigonometric form by solving (2.2)

$$\cos(2\pi v) = \frac{a}{|z|} \Rightarrow$$

$$2\pi v = \cos^{-1}(\frac{a}{|z|}) \equiv \arccos(\frac{a}{|z|}).$$

The *period* (some authors write periodicity) is defined as the inverse of the frequency:

$$period \equiv \frac{1}{v}$$

and gives the number of periods between two peaks. Hence a frequency of 1/2 corresponds to a *period* of 2 time periods for example.

As an example consider the complex pair $z = 0.25 \pm 0.86i$. Hence $|z| = 0.9$, $2\pi v = 1.29$ and $v = 1.29/(2 \cdot 3.14159) = 0.20531$. The period is then

$$period = \frac{1}{0.20531} = 4.8707.$$

Complex roots root play a central role as "drivers of" the solutions of dynamic equations, and of the properties of dynamic multipliers. Hence, if the pair $z = 0.25 \pm 0.86i$ are the roots in the solution of model with second order dynamics, the solution will be cyclical (with dampened cycles) and there will be approximately 5 periods between two peaks. Hence about 1/5 of a cycle is completed in each time period. (e.g., a year).

Formally, the definition of *period* requires $v > 0$. However, in practice it creates no misunderstanding to say that the "zero frequency corresponds to an infinite *period*". In fact, many economic time series variables are charaterized by low (estimated) frequenzies, and we speak of *low frequency data* and *long-memory processes* as typical features in economics.

4 The exponential function

A function of a complex number is referred to as a *complex function of a complex variable.* In this course we will need to the complex exponential function.

Definition 5 (The exponential function) *If z is a complex number*

$$z = x + iy$$

we have that

$$\exp(z) \stackrel{def}{=} \exp(x)(\cos y + i \sin y)$$

which is the definition of the (the natural) exponential function for complex values of z.

In most cases the complex exponential function have the same properties as the real version of the function (except when expressions involve $>$ or $<$). For example we have

$$\exp(z_1)\exp(z_2) = \exp(z_1 + z_2) \tag{4.3}$$

since

$$\exp(x_1)\{\cos(y_1) + i\sin(y_1)\} \cdot \exp(x_2)\{\cos(y_2) + i\sin(y_2)\} =$$
$$\exp(x_1 + x_1)\{\cos(y_1 + y_2) + i\sin(y_1 + y_2)\} = \exp(z_1 + z_2).$$

However, the complex exponential function also has certain unique properties:

$$\overline{\exp(z)} = \exp(\overline{z}) \tag{4.4}$$

and

(4.5) $$\exp(z + ik2\pi) = \exp(z) \quad k = 0, \pm 1, \pm 2, ..$$

(4.5) follows from (4.3) together with

(4.6) $$\exp(ik2\pi) = \cos(k2\pi) + i\sin(k2\pi) = 1$$

since sin and cos have the same *period* 2π.

Definition 6 (Exponential form) *From the trigonometric representation we have that*

$$z = a + ib = |z|(\cos\theta + i\sin\theta).$$

From the definition of the exponential function, we see that

$$(\cos\theta + i\sin\theta) = \exp(i\theta)$$

implying that another way of writing the complex number z is

(4.7) $$z = a + ib = |z|\exp(i\theta).$$

The following rules are useful:

(4.8) $$\begin{array}{lcl} \exp(ix) & = & \cos(x) + i\sin(x) \\ \exp(-x) & = & \cos(x) - i\sin(x) \\ \cos(x) & = & \{\exp(ix) + \exp(-ix)\}/2 \\ \sin(x) & = & \{\exp(ix) - \exp(-ix)\}/2i \end{array}$$

5 The unit circle

The so called *unit circle*, see Hamilton (1994, p 709), can be defined with the use of this representation.

Definition 7 *The complex unit circle is defined as the set of complex numbers that have norm (or modulus) equal to one* 1.

We say that $z = a + ib = |z|\exp(i\theta)$ in on the unit circle when $|z| = 1$. z is inside the unit circle when $|z| < 1$, and finally, that z is outside the unit circle when $|z| > 1$.

The complex number given by

(5.9) $$z = \exp(iy) = \cos(y) + i\sin(y)$$

defines the unit circle when $0 \leq y \leq 2\pi$, since $|\exp(iy)| = \sqrt{\cos(y)^2 + \sin(y)^2} = 1$ from the properties of *sin* and *cos*; or directly:

$$|\exp(iy)| = \sqrt{|\exp(iy)|^2} = \sqrt{\exp(-iy)\exp(iy)} = \sqrt{\exp(0)} = 1.$$

In *spectral analysis* we will define a variable $0 \leq v \leq 1$ called the *frequency* which measures the number of cycles per unit of time. With reference to (5.9) we set $y = 2\pi v$. This gives the following listing of the relationship between frequency and points on the unit circle:

	$\cos(2\pi v)$	$\sin(2\pi v)$	z	$\|z\|$
$v = 0$	1	0	1	1
$v = 1/4$	0	1	i	1
$v = 1/2$	-1	0	-1	1
$v = 3/4$	-1	-1	$-i$	1

6 The fundamental theorem of algebra

We have from, for example Sydsæter (1978, Ch. 12) or **?**, p 114, that:

Theorem 1 (The Fundamental Theorem of Algebra) *Any polynomial of degree p can be factorized into factors of degree 1:*

$$p(x) = a_0x^n + a_1x^{n-1} + ... + a_{p-1}x + a_p \tag{6.10}$$
$$= a_0(x - r_1)^{m_1}(x - r_2)^{m_2}...(x - r_l)^{m_2} \tag{6.11}$$

where multiplicities m_j satisfy:

$$m_1 + m_2 + ... + m_l = p.$$

$r_1, r_2, ...r_l$ are roots of the homogenous n'th order equation:

$$a_0x^p + a_1x^{p-1} + ... + a_{p-1}x + a_p = 0.$$

In general the roots are (or can be written as) complex numbers. If each r_l is counted m_j times, it follows that at any equation of degree p has n roots.

Theorem 2 (Complex pairs) *If r is a root in an p'th order equation with real coefficents a_i, it follows that that also the conjugate, $\overline{r}$, is a root. This implies that complex roots always come in pairs.*

Proof 1 (Complex pairs) *From the rules for complex numbers:*

$$\begin{aligned}&\overline{a_0r^p + a_1r^{n-1} + ... + a_{p-1}r + a_p} = 0 \Rightarrow\\ &\overline{a_0r^p + a_1r^{p-1} + ... + a_{p-1}r + a_p} = \overline{0} = 0 \Rightarrow\\ &a_0\overline{r}^p + a_1\overline{r}^{p-1} + ... + a_{p-1}\overline{r} + a_p = 0\end{aligned}$$

It follows that if p is odd there must be at least one real root.

In particuar we have that the p'th order equation:

$$X^p = c \tag{6.12}$$

has p roots: We first express c with the use of the exponential form $c = |c| \exp(i(\theta + k2\pi))$, $k = 0, \pm1, \pm2, ..$, where we have used that $\exp(k2\pi i) = 1$. We can then write: (6.12) as:

$$X = \sqrt[n]{|c|}\exp(i\frac{(\theta + k2\pi)}{n})$$

and the solution is found by choosing k such that $0 \leq (\theta + k2\pi)/n < 2\pi$. Specifically we have that

$$X^3 = 1 \Rightarrow X = \exp(i\frac{k2\pi}{3})$$

which gives the roots:

$$\begin{array}{lllll} X & = & \exp(i \cdot 0) & = & 1 \\ X & = & \exp(i\frac{2\pi}{3}) & = & \cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6}) & = & -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ X & = & \exp(i\frac{4\pi}{3}) & = & \cos(\frac{2\pi}{6}) + i\sin(\frac{2\pi}{6}) & = & -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{array}$$

where we use the trigonometric form and that complex roots always come in pairs.

7 Complex functions of a real variable

f is a complex function of a real variable x if f can be written

$$\mathrm{f}(x) = \mathrm{u}(x) + i\,\mathrm{v}(x)$$

where u and v are usual real functions. Derivation and integration are defined in a natural way:

Definition 8 (Derivation and integration)

$$\mathrm{f}'(x) \stackrel{def}{=} \mathrm{u}'(x) + i\,\mathrm{v}'(x)$$

$$\int_a^b \mathrm{f}(x)dx \stackrel{def}{=} \int_a^b \mathrm{u}(x)dx + i\int_a^b \mathrm{v}(x)dx$$

The usual rules for derivations and integration can be used, here are some examples:

$$\begin{array}{lll} (\mathrm{f}+\mathrm{g})' & = & \mathrm{f}' + \mathrm{g}' \\ (\mathrm{f}\,\mathrm{g})' & = & \mathrm{f}'\,\mathrm{g} + \mathrm{g}'\,\mathrm{f} \\ \left(\frac{\mathrm{f}}{\mathrm{g}}\right)' & = & \frac{\mathrm{f}'\,\mathrm{g} - \mathrm{g}'\,\mathrm{f}}{\mathrm{g}^2} \\ \int(\mathrm{f}+\mathrm{g})dx & = & \int \mathrm{f}\ dx + \int \mathrm{g}\ dx \\ \int_a^b \mathrm{f}'\ dx & = & \mathrm{f}(b) - \mathrm{f}(a) \end{array}$$

Example 1 *Since* $\exp(ix) = \cos(x) + i\sin(x)$ *we have*

$$\begin{array}{lll} \mathrm{f}'(x) & = & -\sin(x) + i\cos(x) \\ & = & i\left[\frac{-1}{i}\sin(x) + \cos(x)\right] \\ & = & i\left[-(-i)\sin(x) + \cos(x)\right] \\ & = & i\exp(ix) \end{array}$$

where we use $\cos'(x) = -\sin(x)$ *and* $\sin'(x) = \cos(x)$ *and* $1/i = -i$ *and the rules in (4.8).*

Exercise 1 *Show that*

$$\int_a^b \exp(ix)dx = \frac{1}{i}(\exp(ib) - \exp(ia))$$

References

Hamilton, J. D. (1994). *Time Series Analysis*. Princeton University Press, Princeton.

Sydsæter, K. (1978). *Matematisk analyse, Bind II*. Universitetsforlaget, Oslo.