Spectral analysis. Reference note to lecture 9 in ECON 5101 Time Series Econometrics

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1 Introduction

This reference note is a self contained supplement to for example Ch 6 in Hamilton's book. The emphasis is on the theoretical population power spectrum, and on the impact of filtering and variable tranformations, on the power spectrum. Specifically we give the background to the famous characterisation of economic time series in terms of a "the typical spectral shape", Granger (1966), see also Granger and Newbold (1986, Ch 2.7).

Spectral analysis is also referred to in the theory of integrated and cointegrated variables, and the jargon generally assumes some familiarity with analysis in the specteral domain, for example "low frequency unit-root".

This note is influenced by the exposition in Schumway (1988). The more recent title by Shumway and Stoffer (2000) has the about same mathematical level.[1]

2 Frequency and period

The cosine function $\cos(x)$ takes all its values for values of x betwen 0 to 2π, i.e., $\cos(x) = \cos(x + 2\pi)$. The sine function has the same property. x is measured in radians, but we can change the unit of measurement by writing $x = \lambda t$, where λ is called the *frequency* and is measured in radians and t is time. $\cos(\lambda t)$ is a periodic function of time, and we have $\cos(\lambda t) = \cos(\lambda t + 2\pi)$.

More flexibility can be added by introducing the amplitude A and the phase φ, as in

$$f(t) = A\cos(\lambda t - \varphi) = a\cos(\lambda t) + b\sin(\lambda t) \tag{2.1}$$

where $a = A\cos(\varphi)$ og $b = A\sin(\varphi)$, from the properties of the cosine of a sum of two variables (λt and $-\varphi$ here). A can be determined as $A = \sqrt{a^2 + b^2}$ og $\varphi = \tan^{-1}(b/a)$.

The *period*, C, is defined as the length in time of one full cycle

$$C = \frac{\varphi + 2\pi}{\lambda} - \frac{\varphi}{\lambda} = \frac{2\pi}{\lambda}.$$

[1]An expostion in Norwegian is found in chapter 8 and 10 in Bårdsen, G. og R. Nymoen: Videregående emner i økonometri, Fagbokforlaget, 2014.

If $C = 2$ years, the number of cycles per year is 1/2. We define *frequency* as *the number of cycles per unit of time* , hence $v = C^{-1}$.

We choose to define frequency as v rather than λ because it is practical and intuitive. Still, there is no strong conventions here, and several leading textbooks and software programs maesure frequency in radians.

The relationship between the two definitons of frequency is:

$$\lambda = 2\pi v.$$

In the following we will study functions of v in the interval $[-1/2, 1/2]$. $v = 1/2$ is called the Nykvist-frequency (also called the folding frequency). It is the highest frequency that we can indentify with the use of discrete observations.

3 Discrete Fourier transformation and the periodogram

Let $\{x_t\}$ denote a time series with observations x_0, $x_2, ..., x_{T-1}$. Heuristically it is interesting to approximate this time series as closely as possible by a linear combination of cosine functions, as suggested by the equation

$$x_t = a_0 + \sum_{j=1}^{P} \{a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t)\} + rest$$

This problem turns out to have a solution (Fourier analyis) leading to a the *discrete Fourier transform* (DFT) of the time series $\{x_t\}$:

$$X(k) = X_C(k) - iX_S(k), \tag{3.2}$$

where $v_k = k/(2T)$; $k = 0, 1, 2, ..T - 1$, and $X_C(k)$ and $X_S(k)$ are called the cosine and sine transformations of $\{x_t\}$:

$$X_C(k) = T^{-1/2} \sum_{t=0}^{T-1} x_t \cos(2\pi v_k t) \tag{3.3}$$

and

$$X_S(k) = T^{-1/2} \sum_{t=0}^{T-1} x_t \sin(2\pi v_k t) \tag{3.4}$$

Since $\cos(2\pi v_k t) - i \sin(2\pi v_k t) = \exp\{-2\pi v_k it\}$, we have that $X(k)$ can be written as:

$$X(k) = X_C(k) - iX_s(k) = T^{-1/2} \sum_{t=0}^{T-1} x_t \exp\{-2\pi v_k it\} \tag{3.5}$$

which define $X(k)$ as complex numbers associated with the frequencies v_k.

An important property of the DFT is that given (3.5), we also have the inverse transformation:

$$\begin{aligned} x_t &= T^{-1/2} \textstyle\sum_{k=0}^{T-1} X(k) \exp\{2\pi v_k it\} \\ &= T^{-1/2} \textstyle\sum_{k=0}^{T-1} X(k)\{\cos(2\pi v_k t) + i \sin(2\pi v_k t)\} \end{aligned} \tag{3.6}$$

which brings out that DFT gives what we hoped for, namely a decomposition of $\{x_t\}$ in terms of cosine waves, with $X(k)$ as weights for the different frequencies.

Since $X(k)$ is complex, the real numbered "contribution" from each frequency is defined as $P_x(k)$:

$$P_x(k) = X(k)\overline{X(k)} = |X(k)|^2 = X_C(k)^2 + X_S(k)^2, \tag{3.7}$$

where $\overline{X(k)}$ is the conjugate and $|X(k)|$ is the norm of $X(k)$. $P_x(k)$ is real and $\sqrt{P_x(k)}$ is proportional to the amplitude to the cosine function with frequency v_k. The plot of $P_x(k)$ against v_k is called the *periodogram.*

The periodogram provides one approach to estimation of the power spectral density function that discuss briefly in the last section below, but se Ch 6 in Hamilton's book for more about this.

4 Infinite Fourier transformation of the ACF

The DFT has properties that are very similar to the general results of Fourier transformations of more general functions a_t defined over, $t = 0, \pm 1, \pm 2, \ldots$

If $\sum_{s=-\infty}^{\infty} |a_s| < \infty$, the *Infinite Fourier transformation*, IFT, of $\{a_s\}$ is defined as

$$A(v) = \sum_{t=-\infty}^{\infty} a_t \exp(-2\pi i v t) \tag{4.8}$$

with the inverse:

$$a_t = \int_{-1/2}^{1/2} A(v) \exp(2\pi i v t) dv. \tag{4.9}$$

Note that v is without a subscript, since v is a continuous frequency in this representation.

A direct application of this result gives the spectral representation of the ACF $R_x(m)$ for a stationary time series x_t:

$$R_x(m) = E[(x_{t+m} - \mu)(x_t - \mu)] \tag{4.10}$$

where $\mu = E[x_t]$. Stationarity means that

$$\sum_{m=-\infty}^{\infty} |R_x(m)| < \infty. \tag{4.11}$$

The IFT to $\{R_x(m)\}$ is then:

$$f_x(v) = \sum_{m=-\infty}^{\infty} R_x(m) \exp(-2\pi i v m) \tag{4.12}$$

and

$$R_x(m) = \int_{-1/2}^{1/2} f_x(v) \exp(2\pi i v m) dv. \tag{4.13}$$

where $f_x(v)$ is called the population power spectrum.

5 Power spectral density function (PSD)

The power spectral density function $f_x(v)$ is unique and real if x_t is real. $f_x(v)$ is also positive and symmetric.

We can therefore write

$$(5.14) \qquad x_t \text{ real} \;\Rightarrow\; f_x(v) = f_x(-v) \;\Rightarrow\; R_x(m) = 2\int_0^{1/2} f_x(v)\exp(2\pi ivm)dv$$

telling us that both $R_x(m)$ and $f_x(v)$ are completely described by the frequencies in the interval $0 \leq v \leq 1/2$.

Note that, by setting $m = 0$ in (4.13) we have that the variance of x_t can be written as

$$(5.15) \qquad \mathsf{Var}[x_t] = R_x(0) = \int_{-1/2}^{1/2} f_x(v)dv$$

showing that $f_x(v)dv$ is the "contribution to the variance from each frequency".

Example 1 *If x_t is white-noise, the autocovariance function is:*

$$R_x(m) = \begin{cases} \sigma^2, & m = 0 \\ 0, & m = \pm 1, \pm 2... \end{cases}$$

(4.12) gives

$$f_x(v) = \sigma^2, \quad -1/2 \leq v \leq 1/2$$

showing that for a white-noise process, the power spectrum density function (PSD) is constant and equal on all frequencies.

A time series which is made up of of components from all frequencies, and where all frequencies contribute equally much to the variance of the series is called *white-noise* in analogy with (white) light where all colours in the spectrum is present.

6 Power spectral density to ARMA-series

We start by giving an important theorem for the relationshiop between an *input* time series x_t and a filtered (*output*) series y_t in the time domain.

Theorem 1 (PSD of ARMA) *Let $\{x_t\}$ be stationary with $E[x_t] = 0$ and autocovariance function $R_x(m)$, and spectral density $f_x(v)$. Let*

$$y_t = \sum_{s=-\infty}^{s=\infty} a_s x_{t-s} = a(L)x_{t-s},$$

where $\sum |a_s| < \infty$, is a filter $a(L) = \sum_{s=-\infty}^{s=\infty} a_s L^s$. Let $A(v)$ denote the IFT for this filter:

$$(6.16) \qquad A(v) = \sum_{s=-\infty}^{\infty} a_s \exp(-2\pi ivs) = a(\exp(-2\pi iv)).$$

$A(v)$ is called the frequency response function. The power spectral density for $\{y_t\}$ is:

$$(6.17) \qquad f_y(v) = |A(v)|^2 f_x(v) = |a(\exp(-2\pi iv)|^2 f_x(v).$$

This relation shows that the power spectrum of the input series is changed by filtering and that the effect of the change is described as a multiplication by the squared magnitude of the frequency response function (6.16) at each frequency v. We sketch a proof for this important theorem. First apply the definition of the autocovariance function to express $R_y(m)$ by the spectral density $f_x(v)$:

$$\begin{aligned} R_y(m) &= \mathsf{E}[y_{t+m}y_t] = \textstyle\sum_{j,k=-\infty}^{\infty} a_j a_k R_x(m-j+k) \\ &= \textstyle\sum_{j,k=-\infty}^{\infty} a_j a_k \int_{-1/2}^{1/2} f_x(v)\exp(2\pi i v(m-j+k))dv \end{aligned}$$

where we have made use of $R_x(m-j+k) = \int_{-1/2}^{1/2} f_x(v)\exp(2\pi i v(m-j+k))dv$ according to the IFT. We can change the places of $\sum$ and $\int$ in this expression to obtain:

$$\begin{aligned} R_y(m) &= \int_{-1/2}^{1/2} \left\{\textstyle\sum_{j,k=-\infty}^{\infty} a_j a_k \exp(-2\pi i v j)\exp(2\pi i v k)\right\}\exp(2\pi i v m) f_x(v)dv \\ &= \int_{-1/2}^{1/2} \left\{\textstyle\sum_{j=-\infty}^{\infty} a_j \exp(-2\pi i v j)\sum_{k=-\infty}^{\infty} a_k \exp(2\pi i v k)\right\}\exp(2\pi i v m) f_x(v)dv \\ &= \int_{-1/2}^{1/2} A(v)\overline{A(v)}\exp(2\pi i v m) f_x(v)dv \\ &= \int_{-1/2}^{1/2} |A(v)|^2 \exp(2\pi i v m) f_x(v)dv. \end{aligned}$$

At the same time, from the IFT

$$R_y(m) = \int_{-1/2}^{1/2} f_y(v)\exp(2\pi i v m)dv$$

where $f_y(v)$ is unique. Hence:

$$f_y(v) = |A(v)|^2 f_x(v).$$

With the aid of this theorem we can find the spectral density function of variables that follow (stationary) ARMA-models. Let $y_t \sim \mathsf{ARMA}[p,q]$, i.e.,

$$\phi(L)y_t = \theta(L)\varepsilon_t, \ \ \varepsilon_t \sim \mathsf{UIN}(0,\sigma^2), \tag{6.18}$$

with $\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p$ and $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + ... + \theta_p L^q$.

Without loss of generality, consider a causal ARMA(p,q), i.e., the characteristic equation $\lambda^p - \phi_1\lambda^{p-1} - \ldots - \phi_p = 0$ has all its roots inside the unit circle.

First set $x_t = \phi(L)y_t$; and second set $x_t = \theta(L)\varepsilon_t$, and use the theorem twice to give:

$$f_x(v) = |\phi(\exp(-2\pi i v)|^2 f_{y,ARMA[p,q]}(v)$$

and

$$f_x(v) = |\theta(\exp(-2\pi i v)|^2 f_\varepsilon(v)$$

so that

$$f_{y,ARMA[p,q]}(v) = \frac{|\theta(\exp(-2\pi i v)|^2}{|\phi(\exp(-2\pi i v)|^2}\sigma^2 \tag{6.19}$$

since $f_\varepsilon(v) = \sigma^2$ for a white-noise process.

6.1 PSD of AR(1)

Let $\phi(L) = 1 - \phi_1 L$ and $\theta(L) = 1$ i (6.18). We have:

$$|\theta(\exp(-2\pi iv)|^2 = 1$$

and

$$\begin{aligned} |\phi(\exp(-2\pi iv)|^2 &= \{1 - \phi_1 \exp(-2\pi iv)\}\{1 - \phi_1 \exp(2\pi iv)\} \\ &= 1 - \phi_1(\exp(-2\pi iv) - \exp(2\pi iv)) + \phi_1^2 \\ &= 1 - 2\phi_1 \cos(2\pi v) + \phi_1^2 \end{aligned}$$

since $\exp(-2\pi iv) + \exp(2\pi iv) = 2\cos(2\pi v)$. Substitution in the general expression (6.19) gives:

$$f_{y,ARMA(1,0)} = \frac{\sigma^2}{1 - 2\phi_1 \cos(2\pi v) + \phi_1^2}. \tag{6.20}$$

Note that $v^* = \min_v[1 - 2\phi_1 \cos(2\pi v) + \phi_1^2] = 0$ when $\phi_1 > 0$ and $0 \leq v \leq 1/2$. The PSD has a peak in $v = 0$ and declines with increasing v until $v = 1/2$. If $\phi_1 < 0$, $v^* = 1/2$ and the spectral density is increasing in v, cf. Granger and Newbold (1986, p. 56).

6.2 PSD of ARMA(2,1)

$y_t \sim \mathsf{ARMA}(2,1)$

$$y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} = \varepsilon_t + \theta_1 \varepsilon_{t-1},$$

has the PSD:

$$f_{y,ARMA[2,1]}(v) = \sigma^2 \frac{1 + \theta_1 2\cos(2\pi v) + \theta_1^2}{1 + \phi_1^2 + \phi_2^2 - \phi_1(1 - \phi_2) 2\cos(2\pi v) - \phi_2 2\cos(4\pi v)}.$$

7 Linear filters, power-shift and phase-shift

Let x_t denote a stationary time series with ACF $R_x(m)$ and PSD $f_x(v)$. Let

$$y_t = \sum_{s=-\infty}^{s=\infty} a_s x_{t-s}, \text{ where } \sum_{s=-\infty}^{s=\infty} |a_s| < \infty.$$

The IFT for the filter $\{a_s\}$ is, as we have seen

$$A(v) = \sum_{s=-\infty}^{\infty} a_s \exp(-2\pi ivs).$$

$A(v)$ is the frequency response function and the filter a_s $(s = 0, \pm 1, \pm 2, ...)$ is often called the "impulse-response function" in the literature. Since $A(v)$ often is complex, it is useful to write $A(v)$ on polar-coordinate form:

$$A(v) = |A(v)| \exp(i\kappa(v))$$

where the norm $|A(v)|$ is called the power-shift and $\kappa(v)$ is called phase-shift. It can be shown that symmetric filters have no phase-shifting effect but that one-sided filters a_s $(s = 0, 1, 2, ...)$ have such an effect.

In the following we concentrate on power-shifts. From (6.17) we have

$$f_y(v) = |A(v)|^2 f_x(v) \tag{7.21}$$

showing that a filter can amplify or weaken certain frequencies in the input series x_t. Filters are often classified as "low-pass" or "high pass" depending on whether high or low frequencies are amplified by the filter.

Let $y_t = \Delta x_t$ which implies $a_0 = 1$, $a_1 = -1$, $a_s = 0$ for other values of s. The IFT gives us:

$$A(v) = \exp(-2\pi iv \cdot 0) - \exp(-2\pi iv) = 1 - \exp(-2\pi iv),$$

and therefore:

$$\begin{aligned} |A(v)|^2 &= A(v)\,\overline{A(v)} = (1 - \exp(-2\pi iv))(1 - \exp(2\pi iv)) \\ &= 1 - \exp(-2\pi iv) - \exp(2\pi iv) + \exp(0) = 2(1 - \cos(2\pi v)) \end{aligned}$$

A plot of $|A(v)|^2 = 2(1-\cos(2\pi v))$ will show a curve that starts in zero and increases in v. If x_t has a root close to 1 a the zero frequency, this roots will be removed from the filtered series: The differenced series will be "more stationary" than the level series itself.

One-sided filters a_s $(s = 0, 1, 2, ...)$ can often be defined recursively:

$$y_t = \sum_{s=0}^{s=\infty} a_s x_{t-s} \tag{7.22}$$

which is the result of repeated substitution in:

$$y_t = \sum_{s=1}^{p} b_s y_{t-s} + x_t + \sum_{s=1}^{q} b_s x_{t-s} \tag{7.23}$$

or more compactly:

$$b(L)y_t = c(L)x_t, \tag{7.24}$$

where $b(L) = 1 - b_1L - ..b_pL^p$ and $c(L) = 1 + c_1L + ..b_qL^q$. Next define

$$z_t = b(L)y_t = c(L)x_t \tag{7.25}$$

Using (6.17) twice, then gives:

$$f_z(v) = |B(v)|^2 f_y(v) = |C(v)|^2 f_x(v)$$

where $B(v)$ and $C(v)$ is the IFT-ene of the two filters. This gives

$$f_y(v) = \frac{|C(v)|^2}{|B(v)|^2} f_x(v)$$

showing that the power-shift of the one-sided filter a_s $(s = 0, 1, 2, ...)$ in (7.22) is given by:

$$|A(v)|^2 = \frac{|C(v)|^2}{|B(v)|^2} \tag{7.26}$$

8 Regression in the spectral domain: the cross-spectrum and coherency

Let $\mathbf{w}_t^{'} = (x_{t1}, x_{t2}, \dots, x_{tp})$ be a weakly stationary vector time series. The autocovariance matrix is defined as

$$\mathbf{R}(m) = \mathsf{E}[(\mathbf{w}_{t+m} - \mu)(\mathbf{w}_t - \mu)^{'}. \tag{8.27}$$

This matrix is not symmetric, but

$$R_{ij}(m) = R_{ji}(-m) \Rightarrow R^{'}(m) = R(-m).$$

If $\sum_{m=-\infty}^{\infty} |R_{ij}(m)| < \infty$ we can use the IFT on all the components in $\mathbf{R}(m)$:

$$f_{ij}(v) = \sum_{m=-\infty}^{\infty} R_{ij}(m) \exp(-2\pi i v m) \tag{8.28}$$

and

$$R_{ij}(m) = \int_{-1/2}^{1/2} f_{ij}(v) \exp(2\pi i v m) dv. \tag{8.29}$$

$f_{ii}(v)$ are usual PSDs (power spectral density functions), while $f_{ij}(v)$ are known as cross-spectral density functions. We have

$$f_{ij}(v) = f_{ji}(v). \tag{8.30}$$

Note that $f_{ii}(v)$ is real for if the time series are real but $f_{ij}(v)$ may be complex (valued). The matrix that contains all $f_{ii}(v)$ and $f_{ij}(v)$ is called the spectral matrix of $\{\mathbf{w}_t\}$. We write this matrix as $\mathbf{f}_{\mathbf{w}}(v)$. With reference to (8.29) we can write:

$$\mathbf{R}(m) = \int_{-1/2}^{1/2} \mathbf{f}_{\mathbf{w}}(v) \exp(2\pi i v m) dv \tag{8.31}$$

where the integral is taken over all the elements in the matrix $\mathbf{R}(m)$.

In the rest of this section we consider the case of two variables ($p = 2$), x_t and y_t. The spectral matrix becomes

$$\mathbf{f}_w(v) = \begin{bmatrix} f_x(v) & f_{xy}(v) \\ f_{yx}(v) & f_y(v) \end{bmatrix}. \tag{8.32}$$

Since $f_{xy}(v)$ may be complex, it can be written as

$$f_{xy}(v) = |f_{xy}(v)|^2 \exp(i\gamma_{xy}(v)). \tag{8.33}$$

With the aid of a theorem called The multiple Cramer-representation, which we will not give here, it is possible to show that $|f_{xy}(v)|$ measures the strength of the relationship between the two periodic components $P_x(v_j)$ and $P_y(v_j)$ in $\{x_t\}$ and $\{y_t\}$. $\gamma_{xy}(v_j)$ measurers the phase-shift. Heuristically, $f_{xy}(v)$ provides a frequency based measure of the linear relationship between y_t and x. In order to give a justification of this claim, we look briefly into the tasks associated with the construction of a filter for x_t:

$$z_t = \sum_{s=-\infty}^{\infty} a_s x_{t-s}$$

that gives the best explanation of y_t in the least square sense, that is

$$\min \mathsf{E}\{y_t - z_t\}^2.$$

Alternatively, we can regard the filter as determined by a disturbance

$$\epsilon_t = y_t - \sum_{s=-\infty}^{\infty} a_s x_{t-s}, \tag{8.34}$$

which is uncorrelated with $\{x_t\}$, that is:

$$\mathsf{E}[\epsilon_t x_{t-k}] = 0 \tag{8.35}$$

(8.34) and (8.35) give

$$R_{yx}(k) = \sum_{s=-\infty}^{\infty} a_s R_x(k-s) \quad k = 0, \pm 1, \pm 2 \dots \tag{8.36}$$

directly. From (8.28) we have a relationship between $f_{yx}(v)$ and $R_{yx}(k)$ which we can make use of. Substitution from (8.36) in (8.28) gives, after some manipulation:

$$f_{yx}(v) = \sum_{s=-\infty}^{\infty} a_s \exp(-2\pi i v s) \cdot \sum_{k=-\infty}^{\infty} \exp(-2\pi i (k-s)) R_x(k-s),$$

the first term on the right hand side is the IFT of $\{a_s\}$, while the second term is the definition of the spectral density $f_x(v)$. This means that we have obtained

$$f_{yx}(v) = A(v) f_x(v), \quad A(v) = \sum_{s=-\infty}^{\infty} a_s \exp(-2\pi i v s). \tag{8.37}$$

Since ϵ_t is uncorrelated with z_t in $y_t = z_t + \epsilon_t$ we can write:

$$f_y(v) = f_z(v) + f_\epsilon(v) \tag{8.38}$$

From:

$$z_t = \sum_{s=-\infty}^{\infty} a_s x_{t-s},$$

it follows that

$$f_z(v) = |A(v)|^2 \, f_x(v),$$

which after substitution in (8.38)

$$f_y(v) = |A(v)|^2 f_x(v) + f_\epsilon(v).$$

In order to define formally the notion of frequency-dependent correlation we now define the *squared coherence function* of y with respect to x:

$$\gamma_{yx}^2 = \frac{f_z(v)}{f_y(v)} = \frac{|A(v)|^2 \, f_x(v)^2}{f_x(v)f_y(v)} = \frac{|f_{yx}(v)|^2}{f_x(v)f_y(v)}$$

where the last equality holds with reference to (8.37).

The aquared coherency is always real, while this is not the case for the cross-s-spectral density. We see that

$$0 \leq \gamma_{yx}^2 \leq 1$$

and $\gamma_{yx}^2 = 1$ if $f_\epsilon(v) = 0$.

9 The spectrum of ARIMA-series—the typical spectral shape

Experience tells us that many non-stationary variables can be modelled as stochastic (local) trends models. Such variables become stationary after differencing (i.e., by use of the filter $1 - L$), are called integrated variables and belong to the class of ARIMA-models. It is important therefore to establish the PSD for this model class.

We found above that the PSD of an AR(1) process, cf. equation (6.20). When $\phi_1 = 1$ the PSD becomes

$$f_{y,RW}(v) = \frac{\sigma^2}{2(1 - \cos(2\pi v))} \tag{9.39}$$

which is infinite near the zero-frequency and declines sharply with increasing frequency v. This means that all the information in the series is located at the low frequencies—the series is dominated by "long waves".

It may be noted that, formally, we are on thin ice here, since spectral analysis assume stationarity in the first place. However, if we abstract from som problems near zero, (9.39) can be interpreted as a spectrum.

We can find the PSD to a general ARIMA$[p, 1, q]$ by using the results in section 6 and 7.

Let $z_t \sim \text{ARMA}[p, q]$:

$$\varphi(L)z_t = \theta(L)\varepsilon_t, \;\; \varepsilon_t \sim \mathsf{UIN}(0, \sigma^2). \tag{9.40}$$

Next, let z_t be the difference of y_t: $z_t = \Delta y_t = (1 - L)y_t$, so that

$$f_z(v) = |A(v)|^2 \, f_y(v) \tag{9.41}$$

where $A(v)$ is the IFT to the filter $a_0 = 1$, $a_1 = -1$, $a_s = 0$ for all other s. From the example of a low-pass filter above we have:

$$|A(v)|^2 = 2(1 - \cos(2\pi v)), \tag{9.42}$$

while section 6 showed the result

$$f_z(v) = \frac{|\theta(\exp(-2\pi i v)|^2}{|\varphi(\exp(-2\pi i v)|^2}\sigma^2. \tag{9.43}$$

Combining (9.41), (9.42) and (9.43) gives the PSD of $y_t \sim \text{ARIMA}[p,1,q]$:

$$f_{y,ARIMA}(v) = \frac{\sigma^2}{2(1-\cos(2\pi v))} \frac{|\theta(\exp(-2\pi i v)|^2}{|\varphi(\exp(-2\pi i v)|^2} \tag{9.44}$$

which we can write as:

$$f_{y,ARIMA[p,1,q]}(v) = f_{y,RW}(v) \cdot f_{z,ARMA[p,q]}(v) \tag{9.45}$$

where $z_t = (1-L)y_t$. Since $f_{z,ARMA[p,q]}(v)$ is finite for all frequencies, the PSD of $\mathsf{ARIMA}[p,q]$ will be dominated by the *random-walk* component $f_{y,RW}(v)$ which is infinite at the zero frequency.

As noted, since economic time series as a rule can be represented by estimation of $\mathsf{ARIMA}[p,q]$ models, we expect to find that empirical PSDs typically have a marked peak at the zero-frequency and maybe smaller peaks located at for example the seasonal frequencies. This shape is referred to as Granger's "typical spectral shape".

10 Seasonal integration

So far, we have assumed that any non-stationarity is due to low frequency components, but with quarterly or monthly series there are other possibilities.

10.1 Seasonally integrated series.

Let y_t be generated by:

$$y_t = -y_{t-1} - y_{t-2} - y_{t-3} + \varepsilon_t, \;\; \varepsilon_t \sim \mathsf{UIN}(0,\sigma^2). \tag{10.46}$$

which we can write as

$$S(L)y_t = \varepsilon_t \tag{10.47}$$

where $S(L) = 1 + L + L^2 + L^3$. We can interpret $S(L)$ as a filter and use the results above to give a characterization of y_t in the spectral domain. We start by defining

$$w_t = S(L)y_t = \varepsilon_t$$

with $f_w(v) = \sigma^2$ and $f_w(v) = |A(v)|^2 f_y(v)$ where $A(v)$ is the IFT to the filter $S(L)$. The implied power-shift is

$$\begin{aligned} |A(v)|^2 &= \{1 + \exp(-2\pi i v) + \exp(-4\pi i v) + \exp(-6\pi i v)\} \\ &\quad \cdot \{1 + \exp(2\pi i v) + \exp(4\pi i v) + \exp(6\pi i v)\} \\ &= 4 + 6\cos(2\pi v) + 4\cos(4\pi v) + 2\cos(6\pi v) \end{aligned}$$

so that the PSD of y_t becomes:

$$f_y(v) = \frac{\sigma^2}{|A(v)|^2} = \frac{\sigma^2}{4 + 6\cos(2\pi v) + 4\cos(4\pi v) + 2\cos(6\pi v)} \tag{10.48}$$

which is infinite at $v = \{0,25, 0,5\}$ and "flat" elsewhere. Since

$$\text{Var}[y_t] = 2\int_0^{1/2} f_y(v)dv$$

we see that the variance to y_t becomes infinite, which is the hallmark of an non-stationary series. By construction we also have that the filter $S(L)$ has a power-shift which is zero for the same frequencies ($v = \{0,25, 0,5\}$), so that a seasonally integrated series becomes stationary by use of this filter.

When we have quarterly data, the Δ_4-operator is often used. For the seasonally integrated process above,(10.47), the filtered series $z_t = (1 - L^4)y_t$ gets the PSD

$$f_{\Delta_4 y}(v) = 2(1 - \cos(2\pi v))\sigma^2, \tag{10.49}$$

since (10.47) implies $\Delta_4 y_t = -S(L)y_{t-1} + \varepsilon_t = \Delta\varepsilon_t$.

Den "seasonally-filtered" series has a PSD which is finite for all frequencies. The graph of this PSD starts in zero and increases in v. Specifically, the PSD is finite for the frequencies 1/2 and 1/4.

10.2 Seasonal Random-Walk.

Replace the generating equation (10.46), by

$$y_t = y_{t-4} + \varepsilon_t \tag{10.50}$$

For this model we have that

$$f_y(v) = \frac{\sigma^2}{2(1 - \cos(8\pi v))} \tag{10.51}$$

which has a graph which becomes infinite at $v = 0, 1/4, 1/2$. $f_{\Delta_4 y}(v)$, on the other hand, is of course flat by construction, as a result of the power-shift of the filter $1 - L^4$, which is zero at the same frequency.

While a simple random-walk has a single unit root located at the zero-frequencies, we need the whole unit-circle to characterize possible unit-roots for more general ARIMA models.

For example, the process in (10.50) has four unit-roots since the characteristic polynomial of $(1 - L^4)$ can be factorized as:

$$(1 - z^4) = (1 - z)(1 + z)(1 + z^2)$$

with four roots: $z_1 = 1$, $z_2 = -1$, $z_3 = i$, $z_4 = -i$. All roots have modulus equal to 1 (for example $|z_3| = |i| = \sqrt{|i|^2} = \sqrt{i\bar{i}} = \sqrt{-1(i^2)} = 1$) which satisfies the equation:

$$z = \exp(i2\pi v) = \cos(2\pi v) + i\sin(2\pi v) \tag{10.52}$$

which describes the unit-circle when $0 \le v \le 1$. If we denote a unit-root by z_j and the corresponding frequency by v_j we have that

$$\{z_j, v_j\} = \{1, 0;\ i, 1/2;\ -1, 1/2;\ -i, 3/4\},$$

For a random-walk we have $\{z_j,v_j\} = \{1, 0\}$, while the seasonally integrated series (10.46) has roots $\{z_j,v_j\} = \{\ i, 1/2;\ -1, 1/2;\ -i, 3/4\}$ since $S(L) = (1 + L)(1 + L^2)$.

11 Estimation

Under mild conditions (which are satisfied by causal ARMA processes) $X_c(v_k)$ and $X_s(v_k)$ from the Discrete Fourier Transform will be asymptotically independent and normally distributed with expectation 0 and variance $1/2 f_x(v)$. Hence we have:

$$2\frac{X_c(k)^2}{f_x(v)} + 2\frac{X_s(k)^2}{f_x(v)} = 2\frac{P_x(v_k)}{f_x(v)} \sim \chi^2(2). \tag{11.53}$$

Therefore

$$\mathsf{E}[P_x(v_k)] = \mathsf{E}[2\frac{P_x(v_k)]}{f_x(v)} \cdot \frac{f_x(v)}{2}] = 2\frac{f_x(v)}{2} = f_x(v), \tag{11.54}$$

is an unbiased estimator of the population PSD. The periodogram does not necessarily give a consistent estimator (Show!).

However, estimators that are based on modifications of the periodogram have been developed, and these modified estimators are consistent and have good properties also for moderate sample sizes. They are implemented in PcGive and other programmes.

It might be notes that the uncertainty is usually be larger at the low frequencies than elsewhere (the “leakage” phenomenon), and this can be a problem for the interpretation of empirical PSDs for economic time series.

Estimators that are based on the periodogram are often called non-parametric estimators.

A direct and parametric approach to estimation, is to estimate a well-specified ARIMA model first, and obtain the power spectrum by using the estimated parameters in the formulaes given above.

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