Note to lecture 7: The ADL model derived from the VAR

The following system is an example of a first order Gaussian VAR for the two time series x_t and y_t:

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} \pi_{10} \\ \pi_{20} \end{pmatrix} + \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{yt} \\ \varepsilon_{xt} \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} \varepsilon_{yt} \\ \varepsilon_{xt} \end{pmatrix} \sim N\left(\mathbf{0}, \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_x^2 \end{pmatrix} \mid y_{t-1}, x_{t-1}\right). \tag{2}$$

$\boldsymbol{\varepsilon}_t$ $(= (\varepsilon_{yt}, \varepsilon_{xt})')$ and $\boldsymbol{\varepsilon}_{t\pm j}$ are independent pairs of random variables, for all j, meaning no auto-correlation.

The conditioning in (2) means that the normal distribution of $\boldsymbol{\varepsilon}_t = (\varepsilon_{yt},\, \varepsilon_{xt})'$ in the VAR is conditional on the history of the system. More generally, because of complicated correlation between $(y_t, x_t)'$ and the lags of this vector, we would expect that in order to "achieve" uncorrelated disturbances, it would be necessary to condition (2) on period $t-2$, and $t-3$ as well. Therefore, in general, we have the *VAR(p)* model that we use as our main reference for VAR based econometric modelling. But it is simplest, in terms of notation, to first consider the conditional model of *VAR(1)*.

Start by writing (1) as

$$y_t = \mu_{y,t-1} + \varepsilon_{yt} \tag{3}$$
$$x_t = \mu_{x,t-1} + \varepsilon_{xt} \tag{4}$$

where $\mu_{y,t-1}$ and $\mu_{x,t-1}$ denote the conditional expectations $\mu_{y,t-1} \equiv E(y_t \mid y_{t-1}, x_{t-1})$ and $\mu_{x,t-1} \equiv E(x_t \mid y_{t-1}, x_{t-1})$:

$$\mu_{y,t-1} = \pi_{10} + \pi_{11} y_{t-1} + \pi_{12} x_{t-1} \tag{5}$$
$$\mu_{x,t-1} = \pi_{20} + \pi_{21} y_{t-1} + \pi_{22} x_{t-1}. \tag{6}$$

y_t and x_t given by (2), (3) and (4) have a joint normal distribution which is conditional on $x_{t-1}$and y_{t-1}. It follows from the properties of the normal distribution that the conditional distribution of y_t given x_t is also normal, with expectation:

$$\begin{aligned} E(y_t \mid x_t, x_{t-1}, y_{t-1}) &= \mu_{y,t-1} - \rho_{xy}\frac{\sigma_y}{\sigma_x}\mu_{x,t-1} + \rho_{xy}\frac{\sigma_y}{\sigma_x} x_t \\ &= \pi_{10} - \frac{\sigma_{xy}}{\sigma_x^2}\pi_{20} + \frac{\sigma_{xy}}{\sigma_x^2} x_t + (\pi_{12} - \frac{\sigma_{xy}}{\sigma_x^2}\pi_{22}) x_{t-1} \\ &+ (\pi_{11} - \frac{\sigma_{xy}}{\sigma_x^2}\pi_{21}) y_{t-1} \end{aligned}$$

where we have used that ρ_{xy}, the correlation coefficient between ε_{xt} and ε_{yt}, is

$$\rho_{xy} = \frac{\sigma_{xy}}{\sqrt{\sigma_x^2}\sqrt{\sigma_y^2}}. \tag{7}$$

We can now define parameters:

$$\phi_0 = \pi_{10} - \frac{\sigma_{xy}}{\sigma_x^2}\pi_{20} \tag{8}$$
$$\phi_1 = \pi_{11} - \frac{\sigma_{xy}}{\sigma_x^2}\pi_{21} \tag{9}$$
$$\beta_0 = \frac{\sigma_{xy}}{\sigma_x^2} \tag{10}$$
$$\beta_1 = \pi_{12} - \frac{\sigma_{xy}}{\sigma_x^2}\pi_{22} \tag{11}$$

and write the conditional expectation as

$$E(y_t \mid x_t, x_{t-1}, y_{t-1}) = \phi_0 + \phi_1 y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1}. \tag{12}$$

Finally, define the conditional disturbance ε_t as

$$\varepsilon_t = y_t - E(y_t \mid x_t, x_{t-1}, y_{t-1}) \tag{13}$$

and write the conditional model equation for y_t as

$$y_t = \phi_0 + \phi_1 y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + \varepsilon_t \tag{14}$$

which is an ADL(1,1) model.

Note that the ADL disturbance can also be written as

$$\varepsilon_t = \varepsilon_{yt} - \frac{\sigma_{xy}}{\sigma_x^2}\varepsilon_{xt} \tag{15}$$

with variance:

$$Var(\varepsilon_t \mid x_{t-1}, y_{t-1}) = \sigma^2 = \sigma_y^2(1 - \rho_{xy}^2). \tag{16}$$

Using the same principles, the *ADL(p,p)* model can be derived from a $VAR(p)$. The slide set to lecture 7 shows examples of different parameterizations of the *ADL(4,4)* model, including the equilibrium correction model (ECM).

Exogeneity and pre-determinedness: Remember that the starting point is the joint distribution of ε_{yt} and ε_{xt} conditional on x_{t-1} and y_{t-1}, cf (2). Therefore:

$$E(\varepsilon_t \mid x_{t-1}, y_{t-1}) = 0$$

so the disturbance of the ADL model is uncorrelated with the conditioning variables of the VAR. But we have also

$$E(\varepsilon_t \mid \varepsilon_{xt}) = 0 \tag{17}$$

and since $\varepsilon_{xt} = x_t - E(x_t \mid x_{t-1}, y_{t-1})$ by definition, (17) can alternatively be expressed as:

$$E(\varepsilon_t \mid x_t, x_{t-1}, y_{t-1}) = 0 \tag{18}$$

showing that ε_t is uncorrelated with **all** the explanatory variables of the model, just like in a standard static regression model. Hence, by conditioning, x_t, x_{t-1}, y_{t-1} are exogenous variables, exactly in the limited sense given by (18).

However

$$E(\varepsilon_{t-j} \mid x_t, x_{t-1}, y_{t-1}) \neq 0 \text{ for } j = 1, 2, \ldots$$

since y_{t-1} must depend on ε_{t-1} and earlier disturbances, as seen by writing (14) with y_{t-1} as the left hand side variable, and substituting y_{t-2}:

$$\begin{aligned} y_{t-1} &= \phi_0 + \phi_1 y_{t-2} + \beta_0 x_{t-1} + \beta_1 x_{t-2} + \varepsilon_{t-1} \\ &= \phi_0(1 + \phi_1) + x\text{-terms} + \varepsilon_{t-1} + \phi_1 \varepsilon_{t-2} + ... + \phi_1^2 y_{t-3} \end{aligned}$$

and so on.

Therefore, y_{t-1} is a *pre-determined* variable in (14). It is correlated with ε_{t-1} and earlier disturbances, but not with ε_t and future disturbances. Specifically, y_t is not an strictly exogenous variable, which would entail independence of past as well as of future distubances.

Can x_t and/or x_{t-1} can be strictly exogenous variables in the ADL model for y_t?

Role of Gaussian VAR assumption: In all important respects, the above remains valid if (2) is replaced by an IID assumption for the VAR disturbance. The only expectation is the equations that maps from the parameters of the normal distribution to the parameter of the ADL. But the parameters of the ADL will still be parameters in a conditional expectation (again, just as in the static/ordinary regression model case).

References

Bårdsen, G og Nymoen (2014) Videregående emner i økonometri

Hendry, D. F. (1995) *Dynamic Econometrics*, Oxford University Press