The New Keynesian Phillips Curve: Solution with 2nd order dynamics in the "forcing variable" and forecasts with breaks.

Ragnar Nymoen, University of Oslo

22 October 2009

Contents

1 **Introduction** 1

2 **Derivation of (6), and of K_{z1} and K_{z2}.** 3

3 **Non-zero mean in z_t.** 7

4 **Forecasting, and break in the z_t process** 8

1 Introduction

The hybrid NPC is given as

$$y_t = \underset{\geq 0}{\gamma_f} E_t[y_{t+1}] + \underset{\geq 0}{\gamma_b} y_{t-1} + \underset{\geq 0}{\beta} z_t + \epsilon_t, \tag{1}$$

where $y_t$is the rate of inflation, $E_t[y_{t+1}]$ is the expected rate if inflation in period $t+1$, given the information available for forecasting at the end of period t. x_t is firms' real marginal costs, more generally the "forcing" variable. ϵ_t is a disturbance term and might be assumed to be white-noise under the usual assumption that the theory $(\gamma_f E_t[y_{t+1}] + \gamma_b y_{t-1})$ is correct. In many applications, notably GG and GGL, the disturbance term is omitted which suggests a stronger interpretation, which is often referred to as the NPC holding in "exact form".

Due to the presence of the lead-in-inflation in the NPC model, and $0 \leq \gamma_f \leq 1$ as as stylized fact, the stable solution for y_t, if it exists, has to be of the forward-looking type.

It is instructive to consider the closed form solution, which can be established for the case of the hybrid NPC in equation (1), and a second order autoregressive process for z_t:

$$z_t = \lambda_0 + \lambda_1 z_{t-1} + \lambda_2 z_{t-2} + \eta_t \tag{2}$$

The absence of lags of y_t in (2) is clearly restrictive, i.e., if x_t is the wage-share, π_{t-1}, should appear by definition, but this issue is not tackled here.

Following Bårdsen et al. (2005, Appendix A.), we obtain the pure forward-looking model for the auxiliary variable y_t',

(3) $$y_t' = y_t - r_1 y_{t-1}$$

as

(4) $$y_t^{'} = \frac{1}{r_2} E_t[y_{t+1}^{'}] + \frac{\beta}{\gamma_f r_2} z_t + \frac{1}{\gamma_f r_2} \epsilon_t,$$

where r_1 and r_2 are the roots of

$$r^2 - \frac{1}{\gamma_f} r + \frac{\gamma_b}{\gamma_f} = 0$$

where we choose

$$r_1 = \frac{1 - \sqrt{1 - 4\gamma_f \gamma_b}}{2\gamma_f}$$

and

$$r_2 = \frac{1 + \sqrt{1 - 4\gamma_f \gamma_b}}{2\gamma_f}.$$

From Blanchard og Kahn (1980) there is a unique stable solution if $r_2 > 1$ and $r_1 < 1$. In passing: In the case of "homogenous" NPC ($\gamma_f + \gamma_b = 1$), the r_1 and r_2 become

$$\begin{aligned} r_1 &= 1 \\ r_2 &= \frac{1 - \gamma_f}{\gamma_f} \end{aligned}$$

which will require $\gamma_f < 0.5$ in order not to contradict the assumption of $r_2 > 1$ on which the solution is based, but see below for a more precise statement.

To find the solution for inflation, y_t, it is more efficient to start directely from (A.21) in Bårdsen et al. (2005, Appendix A.):

(5) $$y_t = r_1 y_{t-1} + \frac{\beta}{\gamma_f r_2} \sum_{i=0}^{\infty} (\frac{1}{r_2})^i E_t z_{t+i} + \frac{1}{\gamma_f r_2} \epsilon_t$$

The closed formed solution for inflation uses the rational expectation solution for $E_t z_{t+i}$. For the case of $\lambda_0 = 0$ in equation (2) the next section shows that the solution is

(6) $$y_t = r_1 y_{t-1} + \frac{\beta}{\gamma_f r_2} K_{z1} z_t + \frac{\beta}{\gamma_f r_2} K_{z2} z_{t-1} + \frac{1}{\gamma_f r_2} \epsilon_t$$

with K_{z1} and K_{z2} given by

$$\begin{aligned}
(7) \qquad K_{z1} &= -\frac{r_2}{r_{z2}-r_{z1}}\left\{\frac{\frac{r_{z1}}{r_2}}{1-\frac{r_{z1}}{r_2}}-\frac{\frac{r_{z2}}{r_2}}{1-\frac{r_{z2}}{r_2}}\right\} \\
&= \frac{1}{1-\frac{1}{r_2}(\lambda_1+\frac{1}{r_2}\lambda_2)} \\
(8) \qquad K_{z2} &= \frac{r_2}{r_{z2}-r_{z1}}\left\{\frac{r_{z2}\frac{r_{z1}}{r_2}}{1-\frac{r_{z1}}{r_2}}-\frac{r_{z1}\frac{r_{z2}}{r_2}}{1-\frac{r_{z2}}{r_2}}\right\} \\
&= \frac{\lambda_2}{r_2}\left\{\frac{1}{1-\frac{1}{r_2}(\lambda_1+\frac{1}{r_2}\lambda_2)}\right\} \\
&= \frac{\lambda_2}{r_2}K_{z1}
\end{aligned}$$

where r_{zi} are the roots of the characteristic polynomial associated with (2).

The case of $\lambda_0 \neq 0$ (non-zero mean) is considered in section 3.

2 Derivation of (6), and of K_{z1} and K_{z2}.

The characteristic equation associated with the homogenous part of the difference equation (2) is

$$r_z^2-\lambda_1 r_z-\lambda_2=0$$

with roots:

$$\begin{aligned}
r_{z1} &= \tfrac{1}{2}\lambda_1-\tfrac{1}{2}\sqrt{\lambda_1^2+4\lambda_2} \\
r_{z2} &= \tfrac{1}{2}\lambda_1+\tfrac{1}{2}\sqrt{\lambda_1^2+4\lambda_2}
\end{aligned}$$

Assume that both are less than one in magnitude, so z_t is $I(0)$ stationarity (since we here abstract from location shifts/deterministic regime shifts).

Some useful relationships between the roots:

$$\begin{aligned}
r_{z1}^2 &= \frac{1}{4}\lambda_1^2-\frac{1}{2}\lambda_1\sqrt{\lambda_1^2+4\lambda_2}+\frac{1}{4}(\lambda_1^2+4\lambda_2) \\
r_{z2}^2 &= \frac{1}{4}\lambda_1^2+\frac{1}{2}\lambda_1\sqrt{\lambda_1^2+4\lambda_2}+\frac{1}{4}(\lambda_1^2+4\lambda_2) \\
&= \frac{1}{2}\lambda_1^2+\frac{1}{2}\lambda_1\sqrt{\lambda_1^2+4\lambda_2}+\lambda_2 \\
r_{z2}^2-r_{z1}^2 &= \lambda_1\sqrt{\lambda_1^2+4\lambda_2} \\
r_{z2}-r_{z1} &= \sqrt{\lambda_1^2+4b_{x2}} \\
r_{z1}+r_{z2} &= \lambda_1 \\
r_{z1}r_{z2} &= -\lambda_2
\end{aligned}$$

To find the closed form solution for y_t, we need the mathematical expectations, conditional on period t, from the reduced form of z_t, that is:.

$$E_t z_t = z_t \tag{9}$$
$$E_t z_{t+i} = c_1 r_{z1}^{i+1} + c_2 r_{z2}^{i+1},\ i = 1, 2, ..$$

where the second line uses the general solution of a deterministic 2nd order difference equation (which applies to (2) since it is without a constant, the modified results for the case of a non-zero intercept are shown at the end). The coefficients c_1 and c_2 are determined by the initial conditions; z_t and z_{t-1}.

$$\begin{aligned} z_{t-1} &= c_1 + c_2 \\ z_t &= c_1 r_{z1} + c_2 r_{z2} \end{aligned}$$

Solve for c_1 and c_2:

$$\begin{vmatrix} 1 & 1 \\ r_{z1} & r_{z2} \end{vmatrix} = r_{z2} - r_{z1}.$$

$$\begin{aligned} c_1 &= \frac{1}{r_{z2} - r_{z1}} \begin{vmatrix} z_{t-1} & 1 \\ z_t & r_{z2} \end{vmatrix} = \frac{r_{z2} z_{t-1} - z_t}{r_{z2} - r_{z1}} \\ &= \frac{r_{z2}}{r_{z2} - r_{z1}} z_{t-1} - \frac{1}{r_{z2} - r_{z1}} z_t \\ c_2 &= \frac{1}{r_{z2} - r_{z1}} \begin{vmatrix} 1 & z_{t-1} \\ r_{z1} & z_t \end{vmatrix} = \frac{z_t - z_{t-1} r_{z1}}{r_{z2} - r_{z1}} \\ &= \frac{-r_{z1}}{r_{z2} - r_{z1}} z_{t-1} + \frac{1}{r_{z2} - r_{z1}} z_t \end{aligned}$$

As a check, note that with the initial values (z_t and z_{t-1}) we get, for $i = 1$:

$$c_1 r_{z1}^2 + c_2 r_{z2}^2 = \lambda_1 z_t + \lambda_2 z_{t-1} = E_t z_{t+1}$$

as anticipated.

The infinite sum $\sum_{i=o}^{\infty} (\frac{1}{r_2})^i E_t z_{t+i}$ which is the all important unknown in (5) can therefore be expressed as

$$\begin{aligned} \sum_{i=o}^{\infty} (\frac{1}{r_2})^i E_t z_{t+i} &= z_t + \sum_{i=1}^{\infty} (\frac{1}{r_2})^i E_t z_{t+i} \\ &= z_t + \sum_{i=1}^{\infty} (\frac{1}{r_2})^i c_1 r_{z1}^{i+1} + \sum_{i=1}^{\infty} (\frac{1}{r_2})^i c_2 r_{z2}^{i+1}. \end{aligned}$$

Since:

$$z_t = c_1 r_{z1} + c_2 r_{z2}$$

we can complete the sums back to $i = 0$:

$$\sum_{i=o}^{\infty} (\frac{1}{r_2})^i E_t z_{t+i} = r_2 \sum_{i=0}^{\infty} (\frac{1}{r_2})^{i+1} c_1 r_{z1}^{i+1} + \sum_{i=1}^{\infty} (\frac{1}{r_2})^i c_2 r_{z2}^{i+1}$$

Consider each of the two right hand side terms, and change index to j so that $j = i + 1$:

$$
\begin{aligned}
r_2 \sum_{j=1}^{\infty} (\frac{1}{r_2})^j c_1 r_{z1}^j &= c_1 \sum_{j=1}^{\infty} (\frac{r_{z1}}{r_2})^j = r_2 c_1 (\frac{1}{1 - \frac{r_{z1}}{r_2}} - 1) \\
&= r_2 \left\{ \frac{r_{z2}}{r_{z2} - r_{z1}} z_{t-1} - \frac{1}{r_{z2} - r_{z1}} z_t \right\} \frac{\frac{r_{z1}}{r_2}}{1 - \frac{r_{z1}}{r_2}}
\end{aligned}
$$

$$
\begin{aligned}
r_2 \sum_{j=1}^{\infty} (\frac{1}{r_2})^j c_2 r_{z2}^j &= r_2 c_2 \frac{\frac{r_{z2}}{r_2}}{1 - \frac{r_{z2}}{r_2}} \\
&= r_2 \left\{ \frac{-r_{z1}}{r_{z2} - r_{z1}} z_{t-1} + \frac{1}{r_{z2} - r_{z1}} z_t \right\} \frac{\frac{r_{z2}}{r_2}}{1 - \frac{r_{z2}}{r_2}}
\end{aligned}
$$

We see that the conditions for convergence of these sums are:

$$
\left| \frac{r_{z1}}{r_2} \right| < 1 \text{ and } \left| \frac{r_{z2}}{r_2} \right| < 1 \tag{10}
$$

showing that $r_2 > 1$ is a sufficient condition.

Collecting terms with z_t and z_{t-1}.

z_t:

$$
\begin{aligned}
& r_2 \left\{ -\frac{1}{r_{z2} - r_{z1}} \frac{\frac{r_{z1}}{r_2}}{1 - \frac{r_{z1}}{r_2}} + \frac{1}{r_{z2} - rz} \frac{\frac{r_{z2}}{r_2}}{1 - \frac{r_{z2}}{r_2}} \right\} \\
= \; & -\frac{r_2}{r_{z2} - r_{z1}} \left\{ \frac{\frac{r_{z1}}{r_2}}{1 - \frac{r_{xz}}{r_2}} - \frac{\frac{r_{z2}}{r_2}}{1 - \frac{r_{z2}}{r_2}} \right\} \\
= \; & K_{z1}
\end{aligned}
$$

z_{t-1}:

$$
\begin{aligned}
& r_2 \left\{ \frac{r_{z2}}{r_{z2} - r_{z1}} \frac{\frac{r_{z1}}{r_2}}{1 - \frac{r_{z1}}{r_2}} - \frac{r_{z1}}{r_{z2} - r_{z1}} \frac{\frac{rz}{r_2}}{1 - \frac{r_{z2}}{r_2}} \right\} \\
= \; & \frac{r_2}{r_{z2} - r_{z1}} \left\{ \frac{r_{z2} \frac{r_{z1}}{r_2}}{1 - \frac{r_{z1}}{r_2}} - \frac{r_{z1} \frac{r_{z2}}{r_2}}{1 - \frac{r_{z2}}{r_2}} \right\} \\
= \; & K_{z2}
\end{aligned}
$$

So that we get (6):

$$
y_t = r_1 y_{t-1} + \frac{\beta}{\gamma_f r_2} K_{z1} z_t + \frac{\beta}{\gamma_f r_2} K_{z2} z_{t-1} + \frac{1}{\gamma_f r_2} \epsilon_t.
$$

The expression for K_{z1} and K_{z2} can alternatively be written as:

$$
\begin{aligned}
K_{z1} &= \frac{-r_2}{r_{z2}-r_{z1}}\left\{\frac{\frac{r_{z1}}{r_2}}{1-\frac{r_{x1}}{r_2}}-\frac{\frac{r_{z2}}{r_2}}{1-\frac{r_{z2}}{r_2}}\right\}\\
&= \frac{-r_2}{r_{z2}-r_{z1}}\left\{\frac{\frac{r_{z1}}{r_2}\left(1-\frac{r_{z2}}{r_2}\right)-\frac{r_{z2}}{r_2}\left(1-\frac{r_{z1}}{r_2}\right)}{\left(1-\frac{r_{z1}}{r_2}\right)\left(1-\frac{r_{z2}}{r_2}\right)}\right\}\\
&= \frac{-r_2}{r_{z2}-r_{z1}}\left\{\frac{\frac{r_{z1}}{r_2}-\frac{r_{z2}}{r_2}}{\left(1-\frac{r_{z1}}{r_2}\right)\left(1-\frac{r_{z2}}{r_2}\right)}\right\}\\
&= \frac{-1}{r_{z2}-r_{z1}}\left\{\frac{-(r_{z2}-r_{z1})}{\left(1-\frac{r_{z1}}{r_2}\right)\left(1-\frac{r_{z2}}{r_2}\right)}\right\}\\
&= \frac{1}{r_{z2}-r_{z1}}\left\{\frac{1}{\left(1-\frac{r_{z1}}{r_2}\right)\left(1-\frac{r_{z2}}{r_2}\right)}\right\}\\
&= \left\{\frac{1}{\left(1-\frac{r_{z1}}{r_2}\right)\left(1-\frac{r_{z2}}{r_2}\right)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\left(1-\frac{r_{z1}}{r_2}\right)\left(1-\frac{r_{z2}}{r_2}\right) &= 1-\frac{1}{r_2}r_{z1}-\frac{1}{r_2}r_{z2}+\frac{1}{r_2^2}r_{z1}r_{z2}\\
&= 1-\frac{1}{r_2}(r_{z1}+r_{z2}-\frac{1}{r_2}r_{z1}r_{z2})\\
&= 1-\frac{1}{r_2}(\lambda_1+\frac{1}{r_2}\lambda_2)
\end{aligned}
$$

here have used that $r_{z1}+r_{z2}=\lambda_1$ and $r_{z1}r_{z2}=-\lambda_2$, so that K_{z1} becomes:

$$K_{z1}=\frac{1}{1-\frac{1}{r_2}(\lambda_1+\frac{1}{r_2}\lambda_2)}$$

(another, less brute force, derivation is to use the companion form and the equation

(1.2.46) on page 20 in Hamilton's book.)

$$
\begin{aligned}
K_{z2} &= \frac{r_2}{r_{z2}-r_{z1}}\left\{\frac{r_{z2}\frac{r_{z1}}{r_2}}{1-\frac{r_{z1}}{r_2}} - \frac{r_{z1}\frac{r_{z2}}{r_2}}{1-\frac{r_{z2}}{r_2}}\right\} \\
&= \frac{r_2\frac{r_{z1}r_{z2}}{r_2}}{r_{z2}-r_{z1}}\left\{\frac{1}{1-\frac{r_{z1}}{r_2}} - \frac{1}{1-\frac{r_{z2}}{r_2}}\right\} \\
&= \frac{r_2\frac{r_{z1}r_{z2}}{r_2}}{r_{z2}-r_{z1}}\left\{\frac{(1-\frac{r_{z2}}{r_2})-(1-\frac{r_{z1}}{r_2})}{\left(1-\frac{r_{z1}}{r_2}\right)\left(1-\frac{r_{z2}}{r_2}\right)}\right\} \\
&= \frac{r_2\frac{r_{z1}r_{z2}}{r_2}}{r_{z2}-r_{z1}}\left\{\frac{-\frac{r_{z2}}{r_2}+\frac{r_{z1}}{r_2}}{\left(1-\frac{r_{z1}}{r_2}\right)\left(1-\frac{r_{z2}}{r_2}\right)}\right\} \\
&= \frac{r_2\frac{r_{z1}r_{z2}}{r_2}}{r_{z2}-r_{z1}}\left\{\frac{-(\frac{r_{z2}}{r_2}-\frac{r_{z1}}{r_2})}{\left(1-\frac{r_{z1}}{r_2}\right)\left(1-\frac{r_{z2}}{r_2}\right)}\right\} \\
&= -r_2\frac{r_{z1}r_{z2}}{r_2^2}\left\{\frac{1}{\left(1-\frac{r_{z1}}{r_2}\right)\left(1-\frac{r_{z2}}{r_2}\right)}\right\} \\
&= \frac{\lambda_2}{r_2}\left\{\frac{1}{1-\frac{1}{r_2}(\lambda_1+\frac{1}{r_2}\lambda_2)}\right\} \\
&= \frac{\lambda_2}{r_2}K_{z1}
\end{aligned}
$$

3 Non-zero mean in z_t.

(11) $$z_t = \lambda_0 + \lambda_1 z_{t-1} + \lambda_2 z_{t-2} + \eta_t$$

In this case the solution for z_t is:

$$
\begin{aligned}
E_t z_t &= z_t \\
E_t z_{t+i} &= c_1 r_{z1}^{i+1} + c_2 r_{z2}^{i+1} + \frac{\lambda_0}{1-\lambda_1-\lambda_2},\ i = 1, 2, ..
\end{aligned}
$$

where the 2nd. line is the complete solution to the inhomogenous equation (11), i.e., the long-run mean $\frac{\lambda_0}{1-\lambda_1-\lambda_2}$ is a particular solution when we assume $1-\lambda_1-\lambda_2 \neq 0$. Set

$$z^p = \frac{\lambda_0}{1-\lambda_1-\lambda_2}$$

$$
\begin{aligned}
z_{t-1} &= c_1 + c_2 + z^p \\
z_t &= c_1 r_{z1} + c_2 r_{z2} + z^p
\end{aligned}
$$

$$
\begin{aligned}
c_1 &= \frac{1}{r_{z2}-r_{z1}}\begin{vmatrix} z_{t-1}-z^p & 1 \\ z_t - z^p & r_{z2}\end{vmatrix} = \frac{r_{z2}(z_{t-1}-z^p)-(z_t-z^p)}{r_{z2}-r_{z1}} \\
&= \frac{r_{z2}}{r_{z2}-r_{z1}}(z_{t-1}-z^p) - \frac{1}{r_{z2}-r_{z1}}(z_t-z^p) \\
c_2 &= \frac{1}{r_{z2}-r_{z1}}\begin{vmatrix} 1 & (z_{t-1}-z^p) \\ r_{z1} & (z_t - z^p)\end{vmatrix} = \frac{(z_t-z^p)-(z_{t-1}-z^p)r_{z1}}{r_{z2}-r_{z1}} \\
&= \frac{-r_{z1}}{r_{z2}-r_{z1}}(z_{t-1}-z^p) + \frac{1}{r_{z2}-r_{z1}}(z_t-z^p)
\end{aligned}
$$

The progression $\sum_{i=o}^{\infty}(\frac{1}{r_2})^i E_t z_{t+i}$ in (5) for y_t now becomes:

$$
\begin{aligned}
\sum_{i=0}^{\infty}(\frac{1}{r_2})^i E_t z_{t+i} &= z_t + \sum_{i=1}^{\infty}(\frac{1}{r_2})^i c_1 r_{z1}^{i+1} + \sum_{i=1}^{\infty}(\frac{1}{r_2})^i c_2 r_{z2}^{i+1} \\
&\quad + \sum_{i=1}^{\infty}(\frac{1}{r_2})^i z^p
\end{aligned}
$$

or

$$
\begin{aligned}
\sum_{i=0}^{\infty}(\frac{1}{r_2})^i E_t z_{t+i} &= x_t + \sum_{i=1}^{\infty}(\frac{1}{r_2})^i c_1 r_{z1}^{i+1} + \sum_{i=1}^{\infty}(\frac{1}{r_2})^i c_2 r_{z2}^{i+1} \\
&\quad + z^p \frac{r_2}{r_2-1}
\end{aligned}
$$

so that (6) can be generalized to:

$$
(12)\;\; y_t = r_1 y_{t-1} + \frac{\beta}{\gamma_f r_2} K_{z1}(z_t - z^p) + \frac{\beta}{\gamma_f r_2} K_{z2}(z_{t-1}-z^p) + \frac{\beta}{\gamma_f}\left(\frac{1}{r_2-1}\right) z^p + \frac{1}{\gamma_f r_2}\epsilon_t.
$$

4 Forecasting, and break in the z_t process

The above solution is derived for the constant parameter, no-break, case. Forecasting, following the logic of the theory, would then be: To forecast y_{T_1+1}, denote it by $y^e_{T_1+1}$, use (12) with y_{T_1}, z_{T_1+1} and z_{T_1} and $\epsilon_{T_1+1} = 0$. The forecast error will of course be $\frac{1}{\gamma_f r_2}\epsilon_{T_1+1}$. The one step ahead forecast for y_{T_1+2}, $y^e_{T_1+2}$, will be formed in the same way, but with the use of y_{T_1+1}, z_{T_1+2} and z_{T_1+1} and $\epsilon_{T_1+2} = 0$.

To make multi-step forecasts, just use the usual forward recursion of (12).

Assume that there is a break in the z_t process in period $T_1 + 1$, so that instead of λ_0, the intercept becomes $\lambda_0 + \lambda_0^*$, giving:

$$
z^{p^*} = \frac{\lambda_0 + \lambda_0^*}{1-\lambda_1-\lambda_2}
$$

The way we have set up the solution, with $E_t z_t = z_t$ in (9), it seems like we are stuck with the assumption that agents know about the break, i.e. $E_{T_1+1}z_{T_1+1} = z_{T_1+1}$, so

$$y^e_{T_1+1} = r_1 y_{T_1} + \frac{\beta}{\gamma_f r_2} K_{z1}(z_{T_1} - z^{p^*}) + \frac{\beta}{\gamma_f r_2} K_{z2}(z_{T_1-1} - z^{p^*}) + \frac{\beta}{\gamma_f}\left(\frac{1}{r_2-1}\right) z^{p^*}$$

and

$$y_{T_1+1} = r_1 y_{T_1} + \frac{\beta}{\gamma_f r_2} K_{z1}(z_{T_1} - z^{p^*}) + \frac{\beta}{\gamma_f r_2} K_{z2}(z_{T_1-1} - z^{p^*}) + \frac{\beta}{\gamma_f}\left(\frac{1}{r_2-1}\right) z^{p^*} + \frac{1}{\gamma_f r_2}\epsilon_{T_1+1}$$

In order to open for the possibility of period $T_1 + 1$ breaks that the agents do not know about, we obtain the rational expectations solution for the model made up of

$$y_t = \gamma_f E_{t-1}[y_{t+1}] + \gamma_b y_{t-1} + \beta E_{t-1}[z_t] + \epsilon_t, \tag{13}$$

and the forcing process (11). Despite the conditioning on $t-1$, much of the derivation goes through as before, in particular we obtain

$$y_t = r_1 y_{t-1} + \frac{\beta}{\gamma_f r_2}\sum_{i=0}^{\infty}(\frac{1}{r_2})^i E_{t-1}z_{t+i} + \frac{1}{\gamma_f r_2}\epsilon_t \tag{14}$$

The expectation for $z_{\,t+i}$ is now formed according to

$$\begin{aligned} E_{t-1}z_{t-1} &= z_{t-1} \\ E_{t-1}z_{t+i-1} &= c_1 r_{z1}^{i+1} + c_2 r_{z2}^{i+1} + z^p,\ i = 1, 2, .. \end{aligned} \tag{15}$$

where c_1 and c_1 are determined by the initial conditions z_{t-1} and z_{t-2}:

$$\begin{aligned} c_1 &= \frac{r_{z2}}{r_{z2} - r_{z1}}(z_{t-2} - z^p) - \frac{1}{r_{z2} - r_{z1}}(z_{t-1} - z^p) \\ c_2 &= \frac{-r_{z1}}{r_{z2} - r_{z1}}(z_{t-2} - z^p) + \frac{1}{r_{z2} - r_{z1}}(z_{t-2} - z^p) \end{aligned}$$

Setting $i = 1$ gives

$$\begin{aligned} c_1 r_{z1}^2 + c_2 r_{z2}^2 &= \lambda_1(z_{t-1} - z^p) + \lambda_2(z_{t-2} - z^p) + z^p \\ &= \lambda_1 z_{t-1} + \lambda_2 z_{t-2} + (-\lambda_1 - \lambda_2 + 1)z^p \\ &= \lambda_1 z_{t-1} + \lambda_2 z_{t-2} + \lambda_0 = E_{t-1}z_t. \end{aligned}$$

The closed form solution corresponding to (12) is

$$y_t = r_1 y_{t-1} + \frac{\beta}{\gamma_f r_2} K_{z1}(z_{t-1} - z^p) + \frac{\beta}{\gamma_f r_2} K_{z2}(z_{t-2} - z^p) + \frac{\beta}{\gamma_f}\left(\frac{1}{r_2-1}\right) z^p + \frac{1}{\gamma_f r_2}\epsilon_t. \tag{16}$$

where the only differences are the appearance of $(z_{t-1} - z^p)$ in the place of $(z_t - z^p)$, and that $(z_{t-2} - z^p)$ replaces $(z_{t-1} - z^p)$. This is because the solution for y_t is fundamentally driven by the expectations process for z_t, so if the expectations about z_t are conditioned on $t - 1$, then the solution reflects that exactly.

Assume again that there is a break in the z_t process in period $T_1 + 1$, so that instead of λ_0 the intercept becomes $\lambda_0 + \lambda_0^*$.

We now have $E_{T_1} z_{T_1+1} = \lambda_1 z_{T_1} + \lambda_2 z_{T_1-1} + \lambda_0$, so the one step-ahead forecast error for z_{T_1+1} is $\lambda_0^* + \eta_{T_1+1}$. The forecast for $y^e_{T_1+1}$ is not affected by the break in the z_t process. However, the solution y_{T_1+1} is also unaffected so the mean expectation error is again zero.

It appears that because the DGP is fundamentally expectations driven in the rational equilibrium case, there will "never" be large forecast errors for y_t (that are due to breaks in the forcing process). In the case of model (13) and (11), there will be clear impacts of breaks in the forecasts for z_t though.

Occurrence of forecast failure's for inflation does not seem to sit well with the claim that "inflation DGPs" are driven by rational expectations.

Of course, if the DGP for y_t is not the rational expectation solution, then everything "opens up", i.e., if we make forecast with the RE solution and compare with outcomes from the not-RE data generating process. For example, if the DGP is instead made up of (for simplicity)

$$y_t = \gamma_b y_{t-1} + \beta z_t + \epsilon_t, \tag{17}$$

$$z_t = \lambda_0 + \lambda_1 z_{t-1} + \lambda_2 z_{t-2} + \eta_t \tag{18}$$

So, in period $T_1 + 1$:

$$\begin{aligned} y_{T_1+1} &= \gamma_b y_{T_1} + \beta z_{T_1+1} + \epsilon_{T_1+1}, \\ z_{T_1+1} &= \lambda_0 + \lambda_0^* + \lambda_1 z_{T_1} + \lambda_2 z_{T_1-1} + \eta_{T_1+1} \end{aligned}$$

so the intercept break will be carried into y_{T_1+1} but not into the forecast, $y^e_{T_1+1}$, appearing in the NPC equation.

References

Bårdsen, G., Ø. Eitrheim, E. S. Jansen og R. Nymoen (2005). *The Econometrics of Macroeconomic Modelling*. Oxford University Press, Oxford.

Blanchard, O. og C. Kahn (1980). The Solution of Linear Difference Models under Rational Expectations. *Econometrica*, *48*, 1305—1310.