Answer notes Question 2, Seminar 6

Question 2 (copied from lecture note about E-G representation theorem)

Consider the model

$$\Delta y_t = (\rho - 1)y_{t-1} + b_0 \Delta x_t + b_1 x_{t-1} + \varepsilon_t, \qquad 0 < \rho < 1$$
$$\Delta x_t = u_t,$$

where ε_t and u_t are independent white-noise processes.

Exercise 1 *1. Show that $y_t \sim I(1)$.*

2. Write the system in AR-form:

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \mathbf{A} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} b_0 u_t + \varepsilon_t \\ u_t \end{pmatrix}$$

where

$$\mathbf{A} = \begin{pmatrix} \rho & b_1 \\ 0 & 1 \end{pmatrix}.$$

3. Show that the characteristic roots of $\mathbf{A}$ are $\lambda_1 = 1$ and $\lambda_2 = \rho$.

4. Show that $\mathbf{A}^*$ in the ECM form

$$\begin{pmatrix} \Delta y_t \\ \Delta x_t \end{pmatrix} = \mathbf{A}^* \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} b_0 u_t + \varepsilon_t \\ u_t \end{pmatrix}$$

has roots $z_1 = 0$ and $z_2 = \rho - 1$.

5. Show that $\mathbf{A}^*$ can be written

$$\mathbf{A}^* = \begin{pmatrix} \rho - 1 \\ 0 \end{pmatrix} (1, \gamma)$$

where $\gamma = b_1/(\rho - 1)$.

6. Show that $\boldsymbol{\beta}' = (1, \gamma)$ is the cointegrating vector. Hint: Show that

$$w_t = \boldsymbol{\beta}' \begin{pmatrix} y_t \\ x_t \end{pmatrix} = y_t + \gamma x_t$$

is a stationary process by multiplying the AR-equation with $\boldsymbol{\beta}'$ from the left.

Answer notes

1. Write the model in term of the levels variables

$$y_t = \rho y_{t-1} + b_0 x_t + (b_1 - b_0)x_{t-1} + \varepsilon_t, \qquad 0 < \rho < 1$$
$$x_t = x_{t-1} + u_t,$$

By direct inspection, x_t is $I(1)$, random-walk. From the first equation, as long as $b_0 \neq 0$ and/or $(b_1 - b_0) \neq 0$, y_t is also $I((1)$.

2. Substitute x_t in the first equation

$$\begin{aligned} y_t &= \rho y_{t-1} + b_1 x_{t-1} + \varepsilon_t + b_0 u_t, \quad 0 < \rho < 1 \\ x_t &= x_{t-1} + u_t, \end{aligned}$$

this system can can be writen as a VAR with explicit matrix notation:

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} \rho & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} b_0 u_t + \varepsilon_t \\ u_t \end{pmatrix} \tag{1}$$

where

$$\begin{pmatrix} \rho & b_1 \\ 0 & 1 \end{pmatrix} = \mathbf{A}.$$

3. The eigenvalues of $\mathbf{A}$ are the roots of the characteristic equation:

$$\begin{aligned} |\mathbf{A}{-}\lambda\mathbf{I}| &= 0 \Longleftrightarrow \\ (\rho - \lambda)(1 - \lambda) &= 0 \Longleftrightarrow \\ \lambda^2 - (1 - \rho)\lambda + \rho &= 0 \end{aligned}$$

$$\begin{aligned} \lambda_1 &= 1 \text{ is a root} \\ \lambda_1 &= \rho \text{ is a root} \end{aligned}$$

4. Subtract y_{t-1} on both sides in the first row, and x_{t-1} in the second row. This gives the VAR in the ECM form in the question, with

$$\mathbf{A}^* = \begin{pmatrix} \rho - 1 & b_1 \\ 0 & 0 \end{pmatrix} = (\mathbf{A} - \mathbf{I}).$$

The associated characteristic equation $\mathbf{A}^*$ can be written as

$$|\mathbf{A}^*{-}z\mathbf{I}| = |\mathbf{A}{-}(1 + z)\mathbf{I}| = 0$$

Since $\lambda_1 = 1$ is an eigenvalue of $\mathbf{A}$, $z_1 = \lambda_1 - 1 = 0$. And since $\lambda_2 = \rho$, $z_2 = \lambda_1 - 1 = \rho - 1$.

5. Since $\mathbf{A}$ corresponds to $\mathbf{\Phi}$ in the lecture note, we can use (4.20) in that note to write $\mathbf{A}$ as:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (1 - \rho) \begin{bmatrix} \beta \\ \delta \end{bmatrix} \begin{bmatrix} \gamma & -\alpha \end{bmatrix}.$$

since ρ is the stable-root in the notation of this exercise. Since, $\mathbf{A}^* = \mathbf{A} - \mathbf{I}$, we have

$$\mathbf{A}^* = (1 - \rho) \begin{bmatrix} \beta \\ \delta \end{bmatrix} \begin{bmatrix} \gamma & -\alpha \end{bmatrix}$$

Since x_t is an independent random walk in this exercise, we can set $\delta = 0$. From (2.4) in the lecture note it then follows that $\gamma\beta = 1$.Therefore, normalization on y in the eigenvector, $\gamma = -1$, implies $\beta = -1$:

$$\mathbf{A}^* = \begin{bmatrix} -(1 - \rho) \\ 0 \end{bmatrix} \begin{bmatrix} 1 & \alpha \end{bmatrix}$$

Finally, replace the lecture note symbol α with the symbol γ in this exercise, and use the $\mathbf{A}^*$ expression in the ECM formulation of the VAR:

$$\begin{pmatrix} \Delta y_t \\ \Delta x_t \end{pmatrix} = \underbrace{\begin{bmatrix} -(1 - \rho) \\ 0 \end{bmatrix}}_{\boldsymbol{\alpha}} \underbrace{\begin{bmatrix} 1 & \gamma \end{bmatrix}}_{\boldsymbol{\beta}'} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} b_0 u_t + \varepsilon_t \\ u_t \end{pmatrix}$$

By comparing the two expressions for $\mathbf{A}^*$

$$\begin{pmatrix} \rho - 1 & b_1 \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} -(1 - \rho) & -(1 - \rho)\gamma \\ 0 & 0 \end{pmatrix}$$

we obtain:

$$\begin{aligned} b_1 &= -(1-\rho)\gamma \\ \gamma &= \frac{b_1}{\rho - 1} \end{aligned}$$

6. Multiply (1) by the line vector $\boldsymbol{\beta}' = \begin{pmatrix} 1 & \gamma \end{pmatrix}$. The left hand side of (1) then becomes:

$$\boldsymbol{\beta}' \begin{pmatrix} y_t \\ x_t \end{pmatrix} = y_t + \gamma x_t \equiv w_t$$

On the right hand side of (1) we first get:

$$\begin{aligned} \boldsymbol{\beta}' \begin{pmatrix} \rho & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} &= \begin{pmatrix} \rho & b_1 + \gamma \end{pmatrix} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} \\ &= \rho y_{t-1} + (b_1 + \gamma) x_{t-1}) \\ &= \rho \left[y_{t-1} + \frac{(b_1 + \gamma)}{\rho} x_{t-1} \right] \\ &= \rho \left[y_{t-1} + \frac{(-(1-\rho)\gamma + \gamma)}{\rho} x_{t-1} \right] \\ &= \rho \left[y_{t-1} + \gamma x_{t-1} \right] = \rho w_{t-1} \end{aligned}$$

Finally: multiplication of the disturbance vector in (1) by $\boldsymbol{\beta}'$ gives another vector of $I(0)$ disturbances. Therefore

$$w_t = \rho w_{t-1} + I(0)\text{-terms}$$

and we see that w_t is $I(0)$ because $-1 < \rho < 1$. Therefore, $\boldsymbol{\beta}'$ is the (transposed) cointegrating vector.