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ECON5160: A note on the Feb. 13 lecture (corrected Feb. 21), + problems

This note supplements Seierstad section 1.1–1.3 on infinite-horizon dynamic programming and the application to portfolio management. The portfolio separation property is covered. Problems for the topic are given.

This note is updated Feb. 21, as I had quite recklessly thrown in negative consumption at the end of an equation, rather than deducting it from current wealth. The correction is middle page 3 (see the margin note). Furthermore, problem # 2 is updated with a nonnegativity assumption, and problem # 3 has become somewhat more general, but a hint is given. Finally, the «independent of t» thing is rephrased not to be confused with stochastic independence, and a few typos are taken care of.

1 Superoptimality and optimality

This section supplements the boundedness resp. «(P)» resp. «(N)» conditions in Seierstad pp. 10–11 with an argument that has some interpretation (but, on the other hand, involves a bit of calculations). Consider a Markovian autonomous problem in the sense of Seierstad section 1.2, with the objective of maximizing wrt. controls u taking values in a given (maybe X_t, V_t-dependent) control region U the sum

$$\mathsf{E}\Big[\sum_{t=0}^{\infty}\alpha^t g(X_t, u_t(X_t, V_t))\Big|X_0, V_0\Big] \qquad \text{where } X_{t+1} = f(X_t, u_t, V_{t+1})$$

where $\alpha \in (0, 1]$ is a given discount factor. To establish both the Bellman equation and a criterion for superoptimality, consider the operator $\mathsf{B} = \mathsf{B}_H = \mathsf{B}_H(x, v; u)$ taking as input a given function $H = H(x, v)$ and returning

$$\mathsf{B}_H(x, v; u) = g(x, u) + \alpha\mathsf{E}_v[H(f(x, u, V), v] - H(x, v)$$

where «E_v» is shorthand notation for $\mathsf{E}_v[h(V)] = \mathsf{E}[h(V_{t+1}|V_t = v]$, where $\mathbf{V}$ is an independent copy of $\mathbf{V}_{t+1}|\mathbf{V}_t$ (the time-invariance of the latter makes this well-defined).

Now assume that for a given function F, we have $\mathsf{B}_H(x, v; u) \le 0$ for all(x, v, u). Then for any sequence $\{(x_i, v_i, u_i)\}_{i=0,1,\dots}$ we have $\alpha^i\mathsf{B}_H(x_i, v_i; u_i) \le 0$. Summing up the first $T + 1$

terms $\alpha^i \mathsf{B}_H(x_i, v_i; u_i)$ we have

$$\begin{aligned}0 &\geq \sum_{i=0}^{T} \alpha^i g(x_i, u_i) + \sum_{i=0}^{T} \alpha^i \big(\mathsf{E}_v[H(f(x_i, u_i, V), v)] - J(x_i, v_i)\big) \\ &= \sum_{i=0}^{T} \alpha^i g(x_i, u_i) - H(x_0, v_0) \\ &\quad + \alpha \cdot \big(\mathsf{E}_v[H(f(x_0, u_0, V), v)] - J(x_1, v_1)\big) + \dots \\ &\quad + \alpha^{T-1} \cdot \big(\mathsf{E}_v[H(f(x_{T-1}, u_{T-1}, V), v)] - J(x_T, v_T)\big) \\ &\quad + \alpha^T \mathsf{E}_v[H(f(x_T, u_T, V), v)].\end{aligned}$$

Now if we choose $x_0 = X_0$ and $x_{i+1} = f(X_i, u_i, V_{i+1})$ – looks clairvoyant, but that is no problem at the moment, we know that the inequality holds everywhere and it will therefore hold for $x_{i+1} =$ whatever $f(x_i, u_i, V_{i+1})$ will turn out to be – then all lines but the first and last will vanish when we apply the ((X_0, V_0)-conditional) expectation. We get

$$\begin{aligned}H(x_0, v_0) \geq \mathsf{E}\Big[\sum_{i=0}^{T} \alpha^i g(X_i, u_i)\Big| X_0 = x_0, V_0 = v_0\Big] \\ + \alpha^T \mathsf{E}\big[H(f(X_T, u_T, V_{t+1}), V_t)\big| X_0 = x_0, V_0 = v_0\big].\end{aligned}$$

So if the second line tends to something nonnegative[1] as T grows – meaning, in economic terms, that the *downside of the infinitely distant future will be discounted away for all controls* – then, assuming that the expected sum converges nicely (an issue I will skip here), we will have $H(x_0, v_0) \geq$ the criterion to be maximized for any u, hence $\geq$ the best we can get. That means that $H(x_0, v_0)$ is *superoptimal.*

If furthermore there is a sequence $\{u_t^*\}$ such that $H(x_0, v_0)$ is in fact attained – i.e., $H(x_0, v_0) = \mathsf{E}\big[\sum_{i=0}^{\infty} \alpha^i g(X_i, u_i^*)\big| X_0 = x_0, V_0 = v_0\big]$ – then it is optimal (for then $H(x_0, v_0) = \mathsf{E}\big[\sum_{i=0}^{\infty} \alpha^i g(X_i, u_i^*)\big| X_0 = x_0, V_0 = v_0\big] \geq \mathsf{E}\big[\sum_{i=0}^{\infty} \alpha^i g(X_i, u_i)\big| X_0 = x_0, V_0 = v_0\big]$ for all controls). This can be shown by showing that for this particular $\{u^*(x, v)\}$ we have $\mathsf{B}_H(x, v, u^*(x, v)) = 0$ – which in the presence of the ≥ 0 inequality for all u is the Bellman equation! – and also that for this particular control the distant future will be discounted away (i.e. the upside too). Under suitable regularity conditions, one may also solve the finite-horizon problem generally for each T, call the value function $J(T; x, v)$, and test the T-limit.

2 Discrete-time portfolio composition

This section repeats and slightly extends the derivation of the investment portfolio model from the lecture. If the value of an investment opportunity # j evolves as

$$S_{t+1}^{(j)} = S_t^{(j)} \cdot (1 + r + V_{t+1}^{(j)})$$

[1] actually, the limit is allowed to diverge as long as the lim inf is ≥ 0

so that if at time t you have $\nu_t^{(j)}$ units worth $w_t^{(j)} = \nu_t^{(j)} S_t^{(j)}$ altogether, these will at time $t+1$ (before any rebalancing) be worth

$$\nu_t^{(i)} S_{t+1}^{(j)} = \underbrace{\nu_t^{(i)} S_t^{(j)}}_{w_t^{(i)}} \cdot (1 + r + V_{t+1}^{(j)})$$

Here, r is supposed to be the risk-free interest rate and the stochastic factor V – whose state space should include both positive and negative numbers in order not to provide a free lunch – is the *excess return*. If we assume there are d such risky investment opportunities $j = 1, \ldots, d$ and one risk-free opportunity $j = 0$ for which $V^{(0)} = 0$, then the total worth $\sum_{j=0}^{d} w^{(j)}$ must equal your total wealth. If we put $\mathbf{w} = [w^{(1)}, \ldots, w^{(d)}]$ (note that $w^{(0)}$ is not included), then with total endowment X, we must have $w^{(0)} = X - \mathbf{w}\mathbf{1}$. Then this self-financing[2] wealth process X will have the dynamics

$$X_{t+1} = \mathbf{w}\mathbf{V}_{t+1} + (1+r)\sum_{j=0}^{d} w_t^{(j)} = X_t(1 + r + \mathbf{u}_t \mathbf{V}_{t+1})$$

where $\mathbf{V} = [v^{(1)}, \ldots, v^{(d)}]$ (again, no zeroth component) and $\mathbf{u} = \mathbf{w}/X$ – the latter implicitly assumes that the market is open only for agents with positive wealth. If the wealth shall finance consumption c – but otherwise be self-financing – we can for each time t choose to consume c_t and invest the rest, yielding the difference equation *Corrected Feb 21*

$$X_{t+1} = (X_t - c_t)(1 + r + \mathbf{u}_t \mathbf{V}_{t+1})$$

3 Portfolio optimization and portfolio separation

Assume that we are to maximize expected utility from consumption extracted from such a process. The simplest setup is to assume that you invest until time T, at which you end the process and withdraw everything. This corresponds to $\sup \mathsf{E}[\Upsilon(X_T)|X_0 = x, V_0 = v]$. Such a problem was solved in class, with *constant relative risk aversion*[3] utility $\Upsilon(y) = \frac{1}{1-\gamma} y^{1-\gamma}$, and under the assumptions that $\mathbf{V}_{t+1}|\mathbf{V}_t$ is stochastically independent of «everything else» including

- X and $\mathbf{u}$ – this is a «small agent» assumption where the agent can take for granted that her own wealth and actions do not affect price changes
- its own past – that is the Markov property (since the $S^{(i)}$ are specified Markov, then X will be Markov too)

[2] which means that no money is put into the portfolio or taken out from it

[3] a.k.a. «isoelastic». «CRRA» is for the Arrow-Pratt measures for risk aversion, $-y\Upsilon''(y)/\Upsilon'(y)$ for the relative and $-\Upsilon''(y)/\Upsilon(y)$ for the absolute risk aversion.

and furthermore invariant under t – this is the autonomy assumption, so that each $\mathbf{V}_{t+1}|\mathbf{V}_t$ can be written as an independent copy of «$\mathbf{V}|\mathbf{v}$».

We have *not* assumed that the vector components of $\mathbf{V}$ be mutually independent, rather (ad hoc) that the market does not admit arbitrage opportunities (that is, free lunches with zero risk) – this means that one cannot eliminate downside risk by any portfolio except the safe one, corresponding to $\mathbf{u} = \mathbf{0}$, which we furthermore assume to be the only such portfolio[4].

The *terminal-wealth* problem with CRRA utility turned out to exhibit *two fund monetary separation* – which is to say that there exists two «funds», one of which is the risk-free (hence «monetary»), so that all agents with this utility function (i.e.: regardless of wealth) can be equally well off by investing in these two funds, as by investing in the entire market. Mathematically, two fund monetary separation means (assuming an optimal solution exists) that there is an optimal solution on the form $\mathbf{w} = h_t(X)\mathbf{b}_t$ where h is real-valued and the $\mathbf{b}_t$ do not depend on X. This $\mathbf{b}$ is then referred to as the «mutual fund». The amount invested in the riskless opportunity will then be $X - h\mathbf{b}\mathbf{1}$. For the CRRA case, we established two fund monetary separation $h_t = X_t$ (we did not find $\mathbf{b}_t$, only proved it not to be depend on X), and the value function turned out to be of the form $K\frac{1}{1-\gamma}x_0^{1-\gamma}$.

[4] This latter assumption is without loss of generality, as an investment opportunity may then be left out of the market

4 Problems

1) (a) Seierstad problem 1.11; (b) Seierstad problem 1.14; (c) Seierstad problem 1.15.

2) Consider the portfolio optimization problem (dynamics and assumptions as above)

$$\sup_{\mathbf{u},c} \sum_{t=0}^{T} A_t \frac{c_t^{1-\gamma}}{1-\gamma}$$

where the control regions are so that $c_t \in [0, x_t]$ and $\mathbf{u}$ so that with probability 1, we have $1 + r + \mathbf{u}_t \mathbf{V}_{t+1} \geq 0$.

(a) Consider the finite-horizon problem $T < \infty$. Establish the two-fund monetary separation property and find a form for the value function. Do not try to find the optimal portfolio explicitly. How crucial is the assumption that $V_{t+1}|V_t$ is the same for all t?

(b) Consider the infinite-horizon problem $T = \infty$, with $A_t = \alpha^t$ for some $\alpha \in (0, 1)$, and do as point (a).
Hint: Seierstad problem 1.14 (cf. example 1.8) might be of some help.

(c) What can you say about point (b) with $\alpha = 1$?

(d) For each of the problems above, what difference would it make if the control region for U were restricted by the additional constraint that $u_t^{(i)} \geq 0$, all $i > 0$?

3) Consider the portfolio optimization problem (dynamics and assumptions as above)[5]

$$\sup_{\mathbf{u},c} \sum_{t=0}^{T-1} -A_t e^{-a_t c_t} - e^{-a_T X_T}$$

with the same control regions as problem 2. Here, $A_t \geq 0$ and $a_t \geq 0$ (all t) are given deterministic parameters.

(a) As problem 2) point (a). Hint: try $\mathbf{u} = \mathbf{w}/x = \mathbf{b}h(x)/x$ with $h = x/(x - c)$, and separate portfolio-optimization from consumption-optimization.

(b) As problem 2) point (d)

(c) What difference would it make if the control region for c were relaxed to $(-\infty, x]$, to $[0, \infty)$ or to $(-\infty, \infty)$?

[5]The negative exponential utility functions are «CARA» – constant absolute risk aversion.