University of Oslo / Department of Economics / NCF 2009–03–22

ECON5160: Problems from March 20, for March 27 and Apr 03

This note gives (I) problems you would want to have seen before the March 27 lecture, and (II) a few problems which are solvable by now, and which you would want to have solved before Tore's lecture on April 3rd.

I. The problems in the first section below will be very useful for the March 27 lecture. Some results given in these problems will be covered a bit quickly on March 27; this is not to say that I will assume that you have actually managed to solve the problems completely by then, but you should be able to understand the exposition and what the «show that» results actually say. Problem 1.4 is probably the hardest.

II. For Tore's lecture the 3rd, you will need to have seen the Ornstein-Uhlenbeck process – actually, you have already «almost» seen it in an example in Friday's lecture, but not quite. The problem in section 2 should prepare you.

1 Problems for March 27

1.1 Preliminaries on Brownian motion

Let $\mathbf{B}(t) = (B_1(t), B_2(t), \ldots, B_n(t))$, where the B_i are iid. standard Brownian motion – this $\mathbf{B}$ is what we call standard n-dimensional Brownian motion.

(a) Let $\boldsymbol{\sigma}$ be a given n-vector. When is $\boldsymbol{\sigma} \cdot \mathbf{B}(t)$ a standard Brownian motion?

(b) Let $\boldsymbol{\sigma}(t)$ be an n-vector-valued *deterministic* function of t. When is $\int_0^t \boldsymbol{\sigma}(s) \cdot \mathrm{d}\mathbf{B}(s)$ a standard Brownian motion?

(c) If you got the same condition as I did, you should be able to generalize the previous point, to the case where $\boldsymbol{\sigma}$ is stochastic but independent of $\mathbf{B}$. (Hint: Double expectation.)

Comment: The common belief that sums of Gaussians are necessarily Gaussian, is too sloppy. Sums of *jointly* Gaussians are, sums of *marginally* Gaussians are not necessarily.

1.2 Self-financing portfolio composition

This problem will enable you to generate self-financing portfolios in frictionless markets. It requires no stochastic calculus except the fact that the ordinary product rule holds when one of the factors is «locally riskless», i.e. a $\mathrm{d}t$-integral.

Assume that we have a frictionless market consisting of $n+1$ investment opportunities whose prices at time t are denoted $Z_i(t)$. Denote your holdings of investment opportunity no. i at time t by $\nu_i(t)$ units. Then the total market value of your portfolio is

$$Y(t) = \nu_0(t)Z_0(t) + \nu_1(t)Z_1(t) + \cdots + \nu_n(t)Z_n(t).$$

Definition: An investment strategy is called *self-financing* if Y satisfies

$$\mathrm{d}Y(t) = \nu_0(t)\,\mathrm{d}Z_0(t) + \nu_1(t)\,\mathrm{d}Z_1(t) + \cdots + \nu_n(t)\,\mathrm{d}Z_n(t)$$

(this has the interpretation that wealth changes are due to price changes only, not to adding or deducting from the portfolio. Also, an investment strategy $Y(t)-C(t)$ is said to finance consumption $C(t)$ if Y is self-financing.

For the rest of this problem, assume Y self-financing, and that the price dynamics may be written as

$$\mathrm{d}Z_i(t) = Z_i(t)\,\mathrm{d}X_i(t)$$

where Z_0 is «locally riskless» in the sense that $\mathrm{d}X_0(t) = r(t)\,\mathrm{d}t$ (consider r the riskless interest rate).

(a) Let $v_i(t)$ denote the market value of your position in opportunity no. i at time t.

$$\mathrm{d}Y(t) = v_0(t)\,\mathrm{d}X_0(t) + v_1(t)\,\mathrm{d}X_1(t) + \cdots + v_n(t)\,\mathrm{d}X_n(t)$$

and that if $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ (no v_0 nor X_0!) and $\mathbf{1}$ is an n-vector of ones, then

$$\mathrm{d}Y(t) = Y(t)\,\mathrm{d}X_0(t) + \mathbf{v}(t) \cdot (\,\mathrm{d}\mathbf{X}(t) - \mathbf{1}\,\mathrm{d}X_0(t))$$

(Hint: $v_0 + \mathbf{v} \cdot \mathbf{1} = Y$.)

(b) Take Z_0 as numéraire, and put

$$\begin{aligned} \hat{Y}(t) &= Y(t)/Z_0(t) \\ \hat{\mathbf{v}}(t) &= \mathbf{v}(t)/Z_0(t) \\ \hat{\mathbf{X}}(t) &= \mathbf{X}(t) - \mathbf{1}X_0(t). \end{aligned}$$

Show that

$$\mathrm{d}\hat{Y}(t) = \hat{\mathbf{v}}(t) \cdot \,\mathrm{d}\hat{\mathbf{X}}(t). \tag{*}$$

1.3 Simple examples on Itô's formula – and why financial mathematics needs it

The headline says it all? OK, I threw in a not-so-simple example for good measure. Throughout this problem, assume X continuous.

First:

(a) In the formula $\mathrm{d}(XY) = Y\,\mathrm{d}X + X\,\mathrm{d}Y + (\mathrm{d}X)(\mathrm{d}Y)$, why is there no «$\frac{1}{2}$» in front of the latter term?

(b) Show that $(B(t))^2 - t$ is a martingale (hint: differentiate).

(c) Let F be given. Find a process $Z(t)$ which can be written as $\mathrm{d}Z = Y\,\mathrm{d}t$ and so that $F(B(t)) - Z(t)$ is a martingale. (Disregard the $\mathsf{E}|X| < \infty$ condition.)

(d) (Last part a bit tricky:) Itô's formula involves the second derivative, and needs to be modified for functions which are not twice continuously differentiable. Consider $A(t) = |B(t)|$.

   i. Show that $\mathsf{E}[A(t+h) - A(t)|A(t)] \geq 0$ for all $h > 0$. (Hint: Jensen's inequality.)

   ii. Show that for some U, $\mathrm{d}A = U(t)\,\mathrm{d}B$ on all intervals where Z does not hit 0.

   iii. Explain why $Z(t) = A(t) - \int_0^t U(s)\,\mathrm{d}B(s)$ must be an increasing process, and why Z only increases when $A = 0$.

Now assume that X is a process which admits the *ordinary chain rule for differentiation* (i.e. no second-order term).

(e) Calculate $\int_0^T (X(t) - X(0))\,\mathrm{d}X(t)$ and show that it is nonnegative.

(f) Calculate $\int_0^T \max\{0, X(t) - X(0)\}\,\mathrm{d}X(t)$.
(Hint: Show that it is 0 if $X(T) \leq X(0)$, and therefore that you can take the lower integration limit to be $T_0 =$ the last time[1] for which $X \leq 0$.)

(g) Consider formula (*) above, where X and v are one-dimensional.

   i. Pick first $v(t) = X(t) - X(0)$. Why is X *useless* for the modeling purpose?

   ii. What about the case $v(t) = \max\{0, X(t) - X(0)\}$?

A comment: Not only does all differentiable X admit the ordinary chain rule in the integrations we performed; it follows (by the definition of the so-called Young integral) that if X is more than *half a time* differentiable, then a version of the above calculations are valid[2] and X is about just as useless for modeling frictionless financial markets.

[1] it does not matter that T_0 is not a stopping time.
[2] Shameless plug: Framstad: *«Arbitrage with the ordinary chain rule»*, Finance Letters 2(6), 2004

1.4 More Itô's formula – exponential martingales and pricing kernels

The relevance of these problems will (hopefully) be clearer after next Friday's lecture; the G thing could be considered a stochastic component of the discount factor, and your result in (d) below is crucial in pricing financial derivatives.

Let Z_0, Z and G satisfy

$$\begin{aligned}
\mathrm{d}Z_0(t) &= r(t)Z_0(t)\,\mathrm{d}t \\
\mathrm{d}Z(t) &= Z(t^-)\big(\mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B(t) + \gamma(t^-)\,\mathrm{d}\tilde{N}\big) \\
&= Z(t^-)\big((\mu(t) - \lambda(t)\gamma(t))\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B(t) + \gamma(t^-)\,\mathrm{d}N\big) \\
\mathrm{d}G(t) &= G(t^-)\big(-\theta(t)\,\mathrm{d}B(t) + \eta(t^-)\,\mathrm{d}\tilde{N}\big) \\
&= G(t^-)\big(-\lambda(t)\eta(t)\,\mathrm{d}t - \theta(t)\,\mathrm{d}B(t) + \eta(t^-)\,\mathrm{d}N\big)
\end{aligned}$$

where the coefficients are possibly stochastic, and where the martingale $\mathrm{d}\tilde{N}(t) = \mathrm{d}N(t) - \lambda(t)\,\mathrm{d}t$, where N is a *time-inhomogeneous* Poisson process with intensity $\lambda(t)$.

(a) First, show that for $\hat{Z}(t) = Z(t)/Z_0(t)$, we have

$$\mathrm{d}\hat{Z}(t) = \hat{Z}(t^-)\big((\mu(t) - r(t) - \lambda(t)\gamma(t))\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B(t) + \gamma(t^-)\,\mathrm{d}N\big)$$

(i.e. the same equation as Z satisfies, but with $\mu - r$ in place of μ.)

(b) Show that (suppressing time-dependence in some of the notation):

$$\mathrm{d}(\hat{Z}(t)G(t)) = \hat{Z}(t^-)G(t^-) \cdot \Big[(\mu - r - (\gamma + \eta)\lambda - \sigma\theta)\,\mathrm{d}t + (\sigma - \theta)\,\mathrm{d}B + (\eta(t^-) + \gamma(t^-) + \eta(t^-)\gamma(t^-))\,\mathrm{d}N\Big]$$

(c) Use this to give a condition on θ and η for $\hat{Z}G$ to be a martingale. (Disregard the $\mathsf{E}|\hat{Z}G| < \infty$ condition.)

(d) Assume from now on that $\eta = 0$ and $\sigma \neq 0$. Then $\theta = \theta_0$ is unique, and if $G(0) = 1$ then G is unique too. With this θ_0, calculate

$$\mathrm{d}(G(t)F(\hat{Z}(t)))$$

where F is a given function. Comment!

(e) Let $\tau \in [t, T]$ be a stopping time. Calculate $\mathsf{E}_t[G(T)F(\hat{Z}(\tau))]$ where E_t means conditional on information available on time t. (Hint: Use the optional sampling theorem, and disregard conditions of technical nature.)

Comment: For each, G defines a formal probability measure Q on events A known at time T, by $\mathsf{Pr}_Q[A] = \mathsf{E}_0[1_A G(T)]$, where actually 1_A – a random variable equalling one if A occurs and 0 otherwise – is an Arrow-Debreu security; if paid out at time T, its price at time 0 in a typical mathematical finance paradigma, can be represented as «$\mathsf{Pr}_Q[A]$ discounted as in the deterministic case» (i.e. with the price today of a zero-coupon bond maturing at T). For this reason, G is often referred to as a *pricing kernel*.

1.5 The generator

In this problem, let $Y(t) = (t, Z(t))$ where $\mathrm{d}Z = Z(t)(\mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B(t))$, the coefficients being deterministic. Let A_Y be the generator of Y.

(a) Let $H(t, z)$ have the form

$$H(t, z) = Kz^q \exp(-\int_0^t \delta(s)\,\mathrm{d}s)$$

where K and q are constant. State a condition on δ for which $\mathsf{A}_Y H = 0$.

(b) Assume constant coefficients. Find a function $F = F(t, z)$ for which

$$\mathsf{A}_Y F(t, z) + z^q e^{-rt} = 0.$$

(Hint: Try $Az^q e^{-rt}$.)

(c) If N is a Poisson process, what is A_N?

2 For April 03

The (mean-reverting) Ornstein-Uhlenbeck process satisfies the stochastic differential equation

$$\mathrm{d}Z = \theta(\mu - Z) + \sigma\,\mathrm{d}B.$$

Assume until the last part below, that the coefficients are constant; notice that with $\mu = 0$, this was solved in an example in the lectures.

(a) Show that

$$Z(t) = \mu + e^{-\theta t}\big[Z(0) - \mu + \int_0^t e^{\theta s}\,\mathrm{d}B(s)\big]$$

in the following three ways: (i) differentiate both sides and verify the differential equation; (ii) use the integrating factor approach; (iii) put $Y(t) = Z(t) - \mu(t)$ and reduce the problem to the case given in the lecture, then substitute back for $Z(t)$ after solving.

(b) Assume first $Z(0)$ a given number. What is the limiting distribution of Z as $t \to \infty$? (Hint: why is Z Gaussian? You might want to check problem 1.1.)

(c) Assume now that $Z(0) = Y$, a stochastic variable drawn independent of B, and $\mathsf{E}[Y^2] < \infty$. Calculate the mean and variance of $Z(t)$. What are their long-term behaviour?

(d) Show that there is one and only one distribution D so that if $Z(0)$ is distributed D, then D is also the limiting distribution.

(e) Allow now time-dependent but deterministic coefficients. Solve the differential equation for $Z(t)$.