ECON5160: EVT problems + solutions

Updated 2009–04–01 with solutions and a correction. This note gives problems for the EVT material (curriculum for those with math 4 credits), fewer than usual. These problems should give you some hints for the EVT part of the term paper.

Consider the characterization that H_ξ is the GEV corresponding to a distribution with cdf $= F$ and tail function $1 - F = \overline{F}$ if and only if for some positive function a we have

corr.: replaced y by u

$$\lim_{u \nearrow x_F} \frac{\overline{F}(u + xa(u))}{\overline{F}(u)} = \overline{G}_{\xi,1} \tag{1}$$

(where $x_F \leq \infty$ is the right endpoint and G is the GPD cdf), and also if and only if

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{U(ty) - U(t)} = \begin{cases} \frac{x^\xi - 1}{y^\xi - 1} & (\text{if } \xi \neq 0) \\ \frac{\ln x}{\ln y} & (\text{if } \xi = 0) \end{cases} \tag{2}$$

where $U(t) = F^{\leftarrow}(1 - 1/t)$, where $F^{\leftarrow}$ is the generalized inverse.

Problems concerning (1)

(a) We know: if $\overline{F}(x) \sim x^{-\alpha}$ as $x \to \infty$, the corresponding GEV is Fréchet with parameter α. Show that this holds if $\overline{F}(x) \sim x^{-\alpha} \ln x$ too. (Hint: don't choose a too complicated.)

(b) What happens if $\overline{F}(x) \sim x^{-\alpha}(\ln x)^k$, some $k > 1$?

Problems concerning (2)

(a) Show that if $\overline{F}(x) = \exp(-v(x))$, then $U(t) = v^{\leftarrow}(\ln t)$.

(b) Consider the behaviour of the Weibull's *left* tail: let X have cdf $F(x) = 1 - \exp(-x^k)$ for $x > 0$, where $k > 0$ is a constant. Calculate U, and show that the corresponding GEV is the Gumbel distribution.

Problems concerning the GPD An insurance company faces iid $G_{\xi,\beta}$ distributed claim sizes («drawn with replacement»), and reinsures against excess over the upper 5% quantile.

(a) What is the distribution of the claims on the reinsurer?

(b) At the end of year 6, the reinsurer receives its 9th claim. Without entering any discussion on formal statistical estimation, what is the natural suggestion for the distribution of the largest claim on the reinsurer in the next 4 years based on a quick and dirty insertation?

Solution to the problems for (1)

(a) Insert $\overline{F}(u) = u^{-\alpha} \ln u$ in the LHS of (1), to obtain

$$\lim_{u \nearrow \infty} \frac{(u + xa(u))^{-\alpha} \ln(u + xa(u))}{u^{-\alpha} \ln u}.$$

Choosing $a(u) = bu$, we get

$$\lim_{u \nearrow \infty} \frac{(u + ubx)^{-\alpha} \ln(u + ux)}{u^{-\alpha} \ln u} = \lim_{u \nearrow \infty} \frac{(1 + bx)^{-\alpha}(\ln u + \ln(bx))}{\ln u} = (1 + bx)^{-\alpha}$$

Now put $b = 1/\alpha$.

(b) The same calculations apply, except with $\lim \left(\frac{(\ln u + \ln(bx))}{\ln u}\right)^k$, which still is 1.

Problems concerning (2)

(a) We want to show the equivalent of $F(U(t)) = 1 - 1/t$, i.e. $t = (\overline{F}(U(t)))^{-1}$. Inserting for the given $\overline{F}$ and the proposed U, the RHS becomes $(\exp(-v(v^{\leftarrow}(\ln t))))^{-1} = t$.

(b) This is as in (a) with $v(x) = x^k$, so that $v^{-1}(y) = y^{1/k}$ and $U(t) = (\ln t)^{1/k}$. Inserting this into the LHS of (2), we get

$$\lim_{t \to \infty} \frac{(\ln t + \ln x)^{1/k} - (\ln t)^{1/k}}{(\ln t + \ln y)^{1/k} - (\ln t)^{1/k}} = \lim_{t \to \infty} \frac{(1 + \frac{\ln x}{\ln t})^{1/k} - 1}{(1 + \frac{\ln y}{\ln t})^{1/k} - 1} = \lim_{t \to \infty} \frac{\frac{\ln x}{k}(1 + \frac{\ln x}{\ln t})^{\frac{1}{k} - 1}}{\frac{\ln y}{k}(1 + \frac{\ln y}{\ln t})^{\frac{1}{k} - 1}} = \frac{\ln x}{\ln y}.$$

Problems concerning the GPD An insurance company faces iid $G_{\xi,\beta}$ distributed claim sizes («drawn with replacement»), and reinsures against excess over the upper 5% quantile.

(a) The reinsurer sees the excess over a given value c, conditioned on being positive. From the handout, p. 165 part (c), this distribution is $G_{\xi,\beta+\xi c}$, where c is the 5% quantile, i.e. the value for which $(1 + c\xi/\beta)^{-1/\xi} = .05$, i.e. $c\xi = \beta \cdot 20^{\xi} - \beta$. So the distribution is $G_{\xi,\beta r}$ where $r = 20^{\xi}$.

(b) Use the handout, p. 165 part (d), with

- $\lambda = 6$, which is the expected number of claims in 4 years if «9 claims in 6 years» is to be interpreted as 3/2 per year;
- ξ as in part (a), and
- β being the scale parameter from part (a), i.e. the β from the original distribution times 20^{ξ}.

Then with the parameters from (a), we get location $= \mu = (80^{\xi} - 20^{\xi})\beta/\xi$ and scale $= \psi = 80^{\xi}\beta$.