University of Oslo / Department of Economics / NCF 2009–04–02

ECON5160: Problems from March 27

Problems from Seierstad: 4.8, 4.10 and 4.11. Problem 4.9 (a) will also be useful.

Problems in consumption–portfolio optimization: Following the argument of the March 22 problem set, we can write a self-financing wealth process Y, discounted with the numéraire Z_0, as $\mathrm{d}Y(t) = \mathbf{v}(t^-) \cdot \mathrm{d}\mathbf{X}(t)$ – the «t^-» in order to allow for jumps in the driving noise $\mathbf{X}$. In order to finance a nondecreasing cumulative consumption process (also discounted by Z_0), we subtract $\mathrm{d}C$ from it; we shall assume that $\mathrm{d}C = c(t)\,\mathrm{d}t$. Hence we assume that our system obeys

$$\mathrm{d}Y(t) = \mathbf{v}(t^-) \cdot \mathrm{d}\mathbf{X}(t) - c(t)\,\mathrm{d}t \qquad \text{with } Y(0) = y\ (> 0),$$

where v_i is the amount of (real) money invested in risky opportunity no. i (freely chosen), and c is the (real) rate of consumption, required to be nonnegative.

We shall always assume that each X_i is a so-called Lévy process, i.e. with stationary independent increments,which roughly means a drift term plus a Brownian part plus a linear combination of Poisson processes. Unless specified otherwise, assume

$$\mathrm{d}X_i = \mu_i\,\mathrm{d}t + \sigma_i\,\mathrm{d}B_i,$$

where the B_i form an independent family (this is for convenience only), and the coefficients are constant.

Consider for a given utility function Υ (concave, increasing) the following criterion to be maximized:

$$\mathsf{E}[\int_0^\tau Ae^{-\delta t}\Upsilon(c(t))\,\mathrm{d}t + 1_{\tau<\infty} \cdot e^{-\delta\tau}\Upsilon(Y(\tau))]$$

where $\tau = \inf\{t \geq 0;\ Y(t) = 0 \text{ or } t \geq T\}$, all constants > 0 unless otherwise specified, and with T possibly $= \infty$ in which case the scrap value is interpreted as zero no matter what the terms following $1_{\tau<\infty}$.

(a) As long as $\Upsilon(0) > -\infty$, point out how you can eliminate τ from the integral and the scrap value by making the control region Y-dependent.

(b) Solve the problem for $T < \infty$, with $\Upsilon(x) = x^\gamma/\gamma$, where $\gamma \in (0, 1)$. (Hint: Write $v_i = u_i Y$, where u_i is the *fraction* invested in no. i, and also $c(t) = k(t)Y(t)$.)

(c) Solve the problem for $T = \infty$, with $\Upsilon(x) = x^\gamma/\gamma$, where $\gamma \in (0, 1)$.

(d) Consider the HJB functional $\tilde{\mathsf{A}}F(y) = -\delta F(y) + \mathsf{A}_Y^{(\mathbf{v},c)} F(y)$. Test the condition $\tilde{\mathsf{A}}F(y) + \ln c \le 0$ for $F(y) = \ln(y+\epsilon)$. Comment?

(e) Assume now that there are Poisson processes too, so that $\mathrm{d}X_i = \mu_i\,\mathrm{d}t + \sigma_i\,\mathrm{d}B + \sum_j \gamma_{ij}\,\mathrm{d}N_{ij}$, where each N_{ij} is a Poisson process, independent of everything else, and with rate λ_{ij}. Write $v_i = u_i Y$ and $c = kY$, and consider the generator A_Y of Y.

- Explain why $\mathsf{A}_Y^{(c,\mathbf{u})} F(y) = \mathsf{A}_0^{(c,\mathbf{u})} F(y) + \sum \lambda_{ij}(F(y\cdot(1+u_i\gamma_{ij})) - F(y))$, where A_0 is what the generator would have been with all the $\lambda_{ij} = 0$.
- What is $\tilde{\mathsf{A}}_Y^{(c,\mathbf{u})} F(y)$ (cf. (d)) when $F(y) = y^\gamma/\gamma$?
- Can you from these calculations say something about the form of the solution of problems (b) and (c) in this new case? What additional restriction do we have to impose on the control region?

A problem in finance (arbitrage-pricing): Recall that if Z is a geometric Brownian motion $\mathrm{d}Z = Z\cdot(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}B)$, then the arbitrage-free price at time $t_0 < T$ of a financial derivative $F(Z(T))$ paid out at T, is $e^{-r(T-t_0)}\mathsf{E}_{t_0}[F(\tilde{Z}(T))]$ where r is the risk-free interest rate, and $\tilde{Z}$ satisfies $\mathrm{d}\tilde{Z} = \tilde{Z}\cdot(r\,\mathrm{d}t + \sigma\,\mathrm{d}B)$, with initial condition $\tilde{Z}(t_0) = Z(t_0)$. Use this to find the arbitrage-free price at time 0 for $(Z(T))^k$ paid out at time T, where k is any real number. Check your answer for $k = 0$ and $k = 1$.