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ECON5160: Theory overview

After a discussion with Tore, we have agreed that you may at the exam expect to pose a model from a text and discuss reasonableness. This means that I retract my preliminary view that the exam problems will necessarily give clearer guidance of what you are supposed to do than e.g. problem 1 for the term paper did.

The modeling examples of T&K, of Schweder's compendium and Espen's notes – not to mention the ones given in lectures and problems – are curriculum in this course, but they are not the scope of this document. This note is only intended to give a rough list of the most essential *theory* covered, and make a few remarks.

T&K I. The basic concepts of probability are preliminaries to this course. For exam purposes, they are tools to be utilized in solving the problems.

T&K II. From ch. II, the random sum material (II.3) serves as an introduction to e.g. compound Poisson processes later on.
The *martingale* theory (II.5) was covered later in the course. You should be able to

- Use the two results due to Doob, namely the maximal inequality and the optional sampling theorem
- Recognize a stochastic integral wrt. a martingale as a martingale (at least modulo the integrability condition)

T&K III. The important part of chapter III – except of course the concepts etc., which are musts – is everything about first-step analysis. This includes the star-marked section III.7 (see also URL in footnote 1), which gives an efficient linear algebra-formulated tool; you are expected to know and use the fundamental matrix concept like in problem III.7.1 (for an idea on what you should understand of infinite-dimensional matrices in this context, see the solution note for this problem).
You will also be expected to be able to use first-step analysis – and also last-step analysis, whenever appropriate – in continuous time.

- Branching processes were not touched upon in the lectures. No knowledge of their specific features will be assumed for the exam.

T&K IV. There are a lot of important concepts and tools in chapter 4. You are for example expected to be able to

- identify a *regular* transition matrix
- identify *stationary distributions* by way of *eigenvalue methods*
- identify *limiting distributions* both by identifying a stationary distribution as limiting, and by way of the basic limit theorem
- classify states
- identify *aperiodic* Markov chains (and also periodic as «not aperiodic», but the period of a Markov chain is not covered in the lectures)
- for not too complicated examples, identify recurent and transient classes as done in section IV.5, but nothing beyond the «cookbook» given in one of the notes[1].

A couple of *recommended problems:* IV.1.3 and IV.4.4. Also, problem IV.3.2 and its solution contain valuable information (note the misprint «commute» for «communicate» in the solution).

★ *Stochastic dynamic programming in discrete time* (Seierstad 1.1–1.3[2] and lectures)
Basically the same as deterministic dynamic programming, but:

- The setup with probabilities (possibly even choice-dependent) may complicate the calculations even though the principles are the same as in the deterministic case;
- You are not expected to check if expectations diverge other than in obvious cases (like $\mathsf{E}\ln(1+uX)$ when you choose u so that uX might be -1 or less)

In finite horizon, you must perform induction proofs. In infinite horizon, you are expected to know and use the Bellman equation; you may also be asked to check the boundedness/(P)/(N) conditions to eliminate «false solutions» (see Seierstad p. 10–11).
Recommended problems: The portfolio selection problems, but not with the details in which the solution says «feinschmecking» or indicates to be beyond the scope of the course.

T&K V. Poisson processes etc.
There are certainly a few probability calculations needed in this chapter and subsequently – the main objective of this chapter is to give preliminaries to section VI.
Nevertheless, you are expected to be able to treat compound Poisson processes as random sums, to perform calculations by conditioning on the Poisson jumps

[1] http://www.uio.no/studier/emner/sv/oekonomi/ECON5160/v09/5160_20090210_note.pdf
[2] Up to and including p. 19 will suffice.

etc. From the problems given, I would certainly hold problem V.6.10 to be too hard.
Skip: V.1.4, V.4.1 – V.5

T&K VI.

- You must, of course, know birth/death-processes, their limiting behaviours in terms of eigenvectors, and processes with absorbing state(s) and their properties in terms of «infinitesimal first-step analysis» of even last-step analysis (section 5).
  For continuous-time discrete state-space Markov chains (not necessarily a birth/death-process), you must know the crucial concept of infinitesimal generator (as in section 6) – and again, stationary distributions in terms of eigenvectors.
- A detail, ref. the term papers: For intensity-driven processes, the intensity itself is the fundamental starting point. If a problem is given in terms of an «intensity», you are not supposed to do the – not the derivation by discrete time and «$o(h)$». You are also expected to be able to manipulate intensities; for example, if the jump intensity at state n is $\lambda(n)$, and there is an independent random variable governing the new state at the jump, say $\mathsf{Pr}[A]$ into states $\in A$, then you are supposed to know that the intensity of a jump into A is $\lambda(n)\mathsf{Pr}[A]$ – and that for disjoint jump sets A and B, jumps into A and jumps into B can be regarded as driven by independent Poisson processes.

Skip: VI.7 and the backwards Kolmogorov equation[3]

★ *Brownian motion*
The course has not focused on T&K's exposition of Brownian motion, rather on Brownian motion as a driving noise in stochastic differential equations (see below). Therefore, a lot of T&K chapter VIII may either be skipped or be considered covered better by Seierstad, Schweder and your lecture notes.
Note however

- VIII.2.1 and 2.2 for some properties of the Brownian motion itself
- VIII.4 on the geometric Brownian motion (see also the mathematical finance part below)

Also, the martingale property of Brownian motion is key.
The #3 most important continuous process in this course – after Brownian motion itself and geometric Brownian motion – is the Ornstein-Uhlenbeck process. However, here we have not followed T&K.

★ *Stochastic calculus, generators and optimal control.* (Seierstad pp. 185–218)
This was covered quite quickly, and the following should be sufficient:

[3] do not confuse this with the «Chapman–Kolmogorov» equation (T&K p. 394), which is treated.

- Be able to apply Ito's formula
- ... and take expectations to get Dynkin's formula – and notice that this works for integrable stopping times too (in particular, recognize as a martingale – modulo integrability – if there is only a $\mathrm{d}B$ integral)
- ... and find the infinitesimal generator of a given process
- Know the Hamilton-Jacobi-Bellman equation in optimal control in terms of an infinitesimal generator (so that you might even have to take as given a generator where you don't know the process it is associated with), including the version for the infinite-horizon problem with discounting where the operator gets subtracted the discount rate times the function.
- Only slight knowledge of stochastic calculus wrt. jump processes. You should however know that if $N(t)$ is a Poisson process with intensity lambda, then $\tilde{N}(t) = N(t) - \lambda t$ is a martingale (and hence $\mathrm{d}\tilde{N}$ integrals are).
- Only slight knowledge of the technical conditions.

Recommended problems in optimal control: recall the two problems from the lecture.

★ *Mathematical finance*
For geometric Brownian motion, you must know that the arbitrage-free pricing by formally altering the probabilities, works by replacing drift with the riskless interest-rate (and then taking ordinary expectation and discount). To the extent that the expectations are «computable», you are expected to be able to actually do this.
Note: This goes *beyond the books* (although T&K formula (4.21) ff uses precisely this trick).

★ *Extreme value theory (only for students with Math 4 credits)*
Core applications: See the problem set from the lecture, and the compulsory term paper problem set, and the solution to both. Most essential page from the handouts is 165.
A notational clarification: the $\sim$ symbol in the EVT exposition means «has the same asymptotic behaviour as» – not «is distributed as». We write $u(x) \sim v(x)$ if $\lim u/v$ exists and is positive.