UNIVERSITY OF OSLO
DEPARTMENT OF ECONOMICS

Home assignment: **ECON5160/9160 – Stochastic Modeling and Analysis**

Handed out: June 5, 2009 at 2:00 p.m.

To be delivered by: June 9, 2009 at 2:00 p.m.

Place of delivery: Department office, 12^{th} floor

Further instructions:

- The questions are in English, but you can give your answers in English, Norwegian, Swedish or Danish.
- The home assignment will be marked and the scale for the mark will be **pass/fail**.
- After completion, please hand in **2 – two –** copies of your paper to the address given above. The papers must not bear your name, but the individual examination number, which you can find in your studentweb.
- In addition, you must fill in the enclosed declaration
- It is of importance that your paper is delivered by the deadline (see above). Papers delivered after the deadline, **will not be corrected**.*)
- All papers must be delivered to the place given above. You must not deliver your paper to the course teacher or send it by e-mail. If you want to hand in your paper **before** the deadline, please contact the department office on 12^{th} floor.

*) If a student believes that she or he has a good cause not to meet the deadline (e.g. illness) she or he should apply for a special arrangement. See http://www.sv.uio.no/english/academics/Special_exam_arrangements.html for further information. Applications for extension must be sent to the Department of Economics. Normally extension will only be granted when there is a good reason backed by supporting evidence (e.g. medical certificate).

Declaration

Please fill in this form and hand it in to the Department of Economics together with your home assignment.

I hereby declare that my **home assignment**, handed in for

Master/PhD.-course in ECON5160/9160 – Stochastic Modeling and Analysis

at Department of Economics, University of Oslo

1. Has not been used for exsams at other educational institutions, in Norway or abroad.
2. Contains no quotations or extracts from written, printed or electronic sources without the source being referred to.
3. All references are listed in the bibliography
4. I am aware the contravention of these rules are a form of cheating, and against the rules of the University.

Oslo, date……………………………………

Students signature: ……………………………………………..

University of Oslo / Department of Economics / 2009–06–05

Exam spring 2009 – ECON5160 (Stochastic Modeling and Analysis)

- There are 4 pages of problems, in addition to the cover note (1 page), the declaration form (1 page) and the attachment on the F-distribution (2 pages).
- Candidates who have Mathematics 4 credits, shall *not* do Problem 2a.
  Candidates who do not have Mathematics 4 credits, shall *not* do Problem 5.
  All other problems apply to all the candidates.
- For further information, refer to the cover note.

Problem 1 (of 5) – *for all candidates*

A discrete-time Markov chain X_n ($n = 0, 1, \dots$) with states $x = 1, 2, \dots$ (note that there is no «zero» state in this problem) has transition matrix which can be written in terms of «matrix blocks» as

$$\mathbf{P} = \begin{bmatrix} \mathbf{J} & \mathbf{0}_{2\times k} & \mathbf{0}_{2\times \ell} \\ \mathbf{0}_{k\times 2} & \mathbf{S} & \mathbf{0}_{k\times \ell} \\ \mathbf{U} & \mathbf{V} & \mathbf{W} \end{bmatrix}$$

where all the boldfaced symbols are matrices. $\mathbf{J} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ (note: *not* the identity). $\mathbf{S}$ is $k \times k$ (where $k \geq 1$ is finite), and the states $3, \dots, k+2$ communicate. $\mathbf{W}$ is $\ell \times \ell$ (where $\ell \geq 1$ may be finite or infinite), and the states $k+3, ...$ communicate. The «$\mathbf{0}$» are the null matrices of the order indicated.

a) Let $Y_n = X_{2n}$. Show that neither X nor Y can have any limiting distribution.

b) Assume that $\mathbf{U} = \mathbf{0}_{\ell\times 2}$ and that and $\mathbf{S}$ is *symmetric*. Let Z be X constrained to states $3, 4, \dots$ (i.e. $Z_t = X_t$ starting at $X_0 = Z_0 > 2$ – then by $\mathbf{U} = \mathbf{0}_{\ell\times 2}$, Z will avoid the two first states $x = 1$ and $x = 2$). Assume furthermore that Z has a stationary distribution $\boldsymbol{\pi}$.

   (i) Find $\boldsymbol{\pi}$ when $\mathbf{V}$ is nonzero,

   (ii) Find $\boldsymbol{\pi}$ when $\mathbf{W} = \left[\begin{smallmatrix} .5 & .5 \\ .5 & .5 \end{smallmatrix}\right]$ (so that $\ell = 2$),

   (iii) For each of the cases (i) and (ii), decide whether $\boldsymbol{\pi}$ is necessesarily a limiting distribution, necessarily not a limiting distribution, or whether our conditions admit examples of either.

(problem 1 continued next page)

For part c) below, assume that $k > 1$ and $\mathbf{S}$ is symmetric, and that $\mathbf{U}$ and $\mathbf{V}$ are *not* both null matrices. A continuous-time birth/death process $R(t)$ with the same state space as X is constructed as follows (you are supposed to take this construction for given):

- At each jump time τ of a Poisson process (independent of everything else) with intensity λ, one observes $r = R(\tau^-)$ and starts the above Markov chain X at $X_0 = r$.
- One then draws a random time ν uniformly on large times $\{a^2, a^2 + 1, \dots, a^2 + a\}$. Then one determines $R(\tau^+)$ as:

$$R(\tau^+) = \begin{cases} R(\tau^-) & \text{if } X_\nu = R(\tau^-) \text{ or } X_\nu \le 2 \\ R(\tau^-) + 1 & \text{if } X_\nu > R(\tau^-) \text{ and } X_\nu > 2 \\ R(\tau^-) - 1 & \text{if } X_\nu < R(\tau^-) \text{ and } X_\nu > 2 \end{cases}$$

Denote by $\delta(r)$ and $\beta(r)$ the limits as $a \to \infty$ of the death and birth rates, respectively, at state r.

c) For a reasonable interpretation of the above construction,

(i) state which $\beta(i)$ and which $\delta(i)$ will necessarily be zero.

(ii) explain why $\beta(3) = \delta(k + 2)$ and find their common value.

Problem 2 (of 5) *– only candidates WITHOUT Mathematics 4 credits shall do part a). All candidates shall do b), c) and d).*

a) Consider the time-homogeneous random walk X_n on the integers $\{\dots, -2, -1, 0, 1, \dots\}$ with transition probabilities $P_{i,i+1} = p = 1 - P_{i,i-1}$ and let τ be the first hitting time of state 0 when $X_0 = 1$, i.e. $\tau = \min\{n > 0;\ X_n = 0\}$.
Show that $\mathsf{Pr}[\tau < \infty] < 1$ when $p > \frac{1}{2}$, that $\mathsf{E}[\tau] = \infty$ when $p = \frac{1}{2}$, and that $\mathsf{E}[\tau] = 1/(1 - 2p)$ otherwise.
(*Hint:* Use that if T_k is the first hitting time of 0 when $X_0 = k$, then $T_k = \tau_1 + \cdots + \tau_k$ where the τ_i are i.i.d. and $\sim \tau$. Then use first-step analysis to derive an equation for $\mathsf{E}[T_k]$.)

b) Imagine a firm with income process I and cost process C, both independent Poisson processes with respective intensities λ_I and λ_C. The wealth at time $t \ge 0$ is then $W(t) = I(t) - C(t)$, with $W(0) = 0$.
Find $\mathsf{E}[W(t)]$, $\mathsf{var}\, W(t)$ and the distribution of $W(t)$ (the latter as an infinite sum, closed form seems not available).

With W as above, and $w \in \{1, 2, 3, \dots\}$, define $V_w = \inf\{t > 0;\ W(t) \ge w\}$.

c) Calculate $\mathsf{E}[V_w]$, and decide when it is finite, by using the results given in part a).

d) Show the result from c) by a more direct method which does not use the results from a). (*Hint:* consider $Y(t) = W(t) - \mathsf{E}[W(t)]$.)

Problem 3 (of 5) *– for all candidates*

A process $X(t) \geq 0$ satisfies the stochastic differential equation

$$\mathrm{d}X(t) = -\frac{k}{2}X(t)\,\mathrm{d}t + \sigma(k)\sqrt{X(t)}\,\mathrm{d}B(t) \tag{X}$$

where B is a standard Brownian motion.

a) Assume in this part that k is a *nonzero* constant and $\sigma(k) > 0$.

   (i) State the generator A_X of X (i.e., state $\mathsf{A}_X f(x)$ for C^2 functions f).

   (ii) Find all functions f in $C^2([0,\infty))$ and with $f' \neq 0$, for which $Y(t) = f(X(t))$ is a martingale; do only check the condition $\mathsf{E}[|Y(t)|] < \infty$ for one of the signs $k < 0$ or $k > 0$ (your choice, one is arguably easier than the other).

b) Consider the optimal control problem

$$V(x) = \sup \mathsf{E}[\int_0^\infty e^{-\delta t} X(t)\sqrt{k(t)}\,\mathrm{d}t | X(0) = x] \tag{C}$$

   where $\delta > 0$ is a constant, and X obeys the differential equation (X) with $k = k(t)$ being our control, to be chosen freely in $[k_0, \infty)$. You can take for granted that the value function V is C^2 with $V' > 0$.

   (i) State the Hamilton–Jacobi–Bellman equation for the problem (C).

   (ii) In the case where $\sigma(k) = \sigma_0$ (constant), find the optimal k^* expressed in terms of the value function V for each of the cases $k_0 = 0$ and $k_0 = 3$.

c) Assume that k_0 is an arbitrary nonnegative constant, and that $\sigma(k)$ is a general function.

   (i) Show that the Hamilton–Jacobi–Bellman equation has a solution $v(x) = \gamma x$, with $\gamma > 0$.

   (ii) Argue that this is the only function solving the Hamilton–Jacobi–Bellman equation, and which is C^2 and positive for $x > 0$.

Problem 4 (of 5) *– for all candidates*

Consider a frictionless market consisting of a safe account accumulating interest at constant rate $r > 0$, and a stock with price evolving according to $\mathrm{d}X(t) = \mu X(t)\,\mathrm{d}t + \sigma X(t)\,\mathrm{d}B(t)$, where B is a standard Brownian motion, μ is a constant and σ has the particular value $\sigma = \sqrt{2r}$. We assume $X(0) = x > 0$.

a) A security pays 1 at time T if $X(T) > x$, and 0 otherwise. Show that its arbitrage-free price at $t = 0$ is $\frac{1}{2}e^{-rT}$.

b) Find the arbitrage-free price at time $t = 0$ of a security that at time T pays 1 if $\max_{t\in[0,T]} X(t) > M$, where $M > x$ is a constant, and 0 otherwise. You may express the answer in terms of the standard Gaussian cumulative distribution function.
(*Hint:* The reflection principle.)

Problem 5 (of 5) *– only candidates with Mathematics 4 credits*

a) Let $V_k = \max\{S_1, \dots, S_k\}$ where the S_i are i.i.d. drawn from the F distribution (see the attachment). What is the generalized extreme value distribution associated to V_k as $k \to \infty$? (I.e.: what is the «ξ» parameter of the GEV, in terms of the degrees of freedom parameters of the F distribution?)

The rest of the problem is unrelated to point a). In the following, all parameters are > 0.

An insurer faces claims arriving at the jump times of a Poisson process $N(t)$ with intensity λ (per year), and (positive) claim sizes Y_i which are i.i.d. with $\mathsf{Pr}[Y_i \leq y] = 1 - (1 + cy)^{-d}$ and independent of N. Let M be next year's largest claim (where $M = 0$ if no claim occurs).

b) Find a, b and ξ so that $aM + b$ is GEV distributed with shape parameter ξ.

c) There are three reinsurance companies. The insurer polls them for premia on reinsurance covering M, and get the following pricing principles:

$$R_1 = \mathsf{E}[M] \cdot (1 + r_1)$$
$$R_2 = \mathsf{E}[M] + r_2\sqrt{\mathsf{E}[M^2]}$$
$$R_3 = F_M^{-1}(r_3) \qquad \text{where } r_3 \in (0, 1) \text{ and } F_M(m) = \mathsf{Pr}[M \leq m]$$

(i) Decide when each R_i is finite.

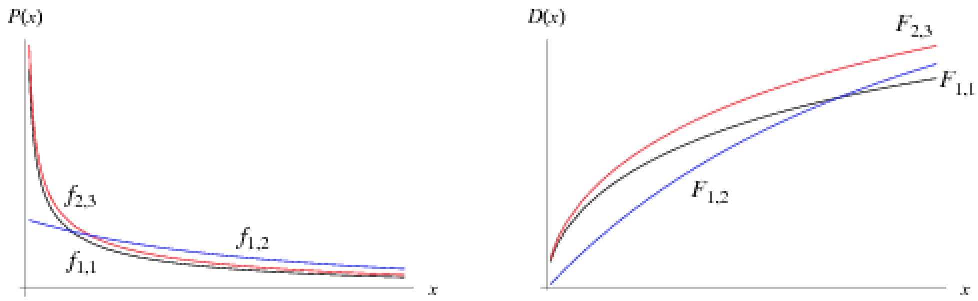
For some parameters, you will find only one of the R_i is finite. Consider this case.

(ii) Describe (in words) what happens to a company which sells this reinsurance for a repeated number of (i.i.d.) years (without buying any protection against the risk), and give economic arguments – from outside this model – why the shareholders could still be willing to let the reinsurance company enter this business strategy.

Probability and Statistics > Statistical Distributions > Continuous Distributions >
Interactive Entries > Interactive Demonstrations >

F-Distribution

DOWNLOAD *Mathematica* Notebook



A continuous statistical distribution which arises in the testing of whether two observed samples have the same variance. Let χ_m^2 and χ_n^2 be independent variates distributed as chi-squared with m and n degrees of freedom.

Define a statistic $F_{n,m}$ as the ratio of the dispersions of the two distributions

$$F_{n,m} \equiv \frac{\chi_n^2 / n}{\chi_m^2 / m}. \qquad (1)$$

This statistic then has an F-distribution on domain $[0, \infty)$ with probability function $f_{n,m}(x)$ and cumulative distribution function $F_{n,m}(x)$ given by

$$f_{n,m}(x) = \frac{\Gamma\left(\frac{n+m}{2}\right) n^{n/2} m^{m/2}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \frac{x^{n/2-1}}{(m+nx)^{(n+m)/2}} \qquad (2)$$

$$= \frac{m^{m/2} n^{n/2} x^{n/2-1}}{(m+nx)^{(n+m)/2} B\left(\frac{1}{2}n, \frac{1}{2}m\right)} \qquad (3)$$

$$F_{n,m}(x) = I\left(\frac{nx}{m+nx}; \frac{1}{2}n, \frac{1}{2}m\right) \qquad (4)$$

$$= 2n^{(n-2)/2}\left(\frac{x}{m}\right)^{n/2} \times \frac{{}_2F_1\left(\frac{1}{2}(m+n), \frac{1}{2}n; 1+\frac{1}{2}n; -nx/m\right)}{B\left(\frac{1}{2}n, \frac{1}{2}m\right)}, \qquad (5)$$

where $\Gamma(z)$ is the gamma function, $B(a, b)$ is the beta function, $I(x; a, b)$ is the regularized beta function, and ${}_2F_1(a, b; c; z)$ is a hypergeometric function.

The F-distribution is implemented in *Mathematica* as `FRatioDistribution`[*n*, *m*].

The mean, variance, skewness and kurtosis are

$$\mu = \frac{m}{m-2} \qquad (6)$$

$$\sigma^2 = \frac{2m^2(m+n-2)}{n(m-2)^2(m-4)} \qquad (7)$$

$$\gamma_1 = \frac{2(m+2n-2)}{m-6}\sqrt{\frac{2(m-4)}{n(m+n-2)}} \qquad (8)$$

$$\gamma_2 = \frac{12\left(-16 + 20\,m - 8\,m^2 + m^3 + 44\,n - 32\,m\,n + 5\,m^2\,n - 22\,n^2 + 5\,m\,n^2\right)}{n\,(m-6)\,(m-8)\,(n+m-2)}. \quad (9)$$

The probability that F would be as large as it is if the first distribution has a smaller variance than the second is denoted $Q\left(F_{n,m}\right)$.

SEE ALSO: Beta Function, Gamma Function, Hotelling T2 Distribution, Noncentral F-Distribution, Regularized Beta Function, Snedecor's F-Distribution

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CITE THIS AS:

Weisstein, Eric W. "F-Distribution." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/F-Distribution.html