1 Introduction to models with heterogenous agents

We will now turn our attention to models without a representative agent. In particular, we will study a class of models – sometimes known as "Bewley-Aiyagari models" – where there is a large cross-section of agents who experience idiosyncratic risk that they cannot trade away through perfect insurance markets.

The Aiyagari-Bewley economic model, proposed by Bewley (1986) and developed further in Aiyagari (1994), has become a workhorse model. The economy is populated with heterogeneous infinitely lived agents subject to uninsurable idiosyncratic income risks. Since agents' histories of income shocks are different, the model generates equilibrium cross-section distributions of wealth, saving and consumption. These cross sectional distributions are contrasted with or calibrated to fit their empirical counterparts in the data, and their responses to various policy changes can be analyzed.

The key elements of the model are

1. There are mass of atomistic individuals, ie. each individuals is a price taker.

2. Agents are *ex-ante* homogeneous but *ex-post* heterogeneous, depending on the history of realizations of shocks to the individual.

3. The asset structure is exogenously determined.

4. Only the steady state equilibrium is studied.

5. Prices (wage and interest rate) are determined competitively.

The outline of the algorithm is:

1. For given factor prices we solve the individuals' optimal decision rules.

2. In the stochastic neoclassical growth model we used the stationary decision rules to approximate the time-series properties of the model. Here we will use the stationary decision rules to approximate the cross-sectional properties of the model.

3. Integrating over the stationary cross-sectional distribution of asset holdings gives us the new aggregate capital stock which will also determine the factor prices.

2 Model

Consider a model economy with a continuum of individuals who each face idiosyncratic employment risk.

2.1 Each individual's problem

Each individual solves the following program

$$\max_{\{c_t,a_{t+1}\}_{t=0}^{\infty}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

subject to

$$c_t + a_{t+1} = \begin{cases} \left[1+(1-\tau)\, r_t\right] a_t + (1-\tau)\, w_t & \text{if } s=e \\ \left[1+(1-\tau)\, r_t\right] a_t + \omega & \text{if } s=u \end{cases}$$

where c is consumption, a is asset holdings, τ is the tax rate, r is rate of return on assets, w is wage income if employed, ω is unemployment benefits, s is employment status, e stands for employed and u stands for unemployed.

Employment status is governed by a discrete two-state Markov process

$$\pi\left(s' \mid s\right) = \Pr\left\{s_{t+1}=s' \mid s_t = s\right\} = \begin{pmatrix} p_{e,e} & p_{e,u} \\ p_{u,e} & p_{u,u} \end{pmatrix}.$$

The individuals solve their dynamic problem

$$v\left(s,a\right) = \max_{a'} \left\{ u\left(c\right) + \beta \sum_{s} \pi\left(s' \mid s\right) v\left(s',a'\right) \right\}$$

The solution is characterized by a unique value function $v\left(s,a\right)$ and a unique stationary decision rule $a'\left(s,a\right)$

2.2 Distributions and aggregates

The return on capital, r_t, the wage rate, w_t, the tax rate, τ_t, and the unemployment benefits, ω_t, each individual faces are independent of her/his choices, but functions of the aggregate stationary distribution of capital and labor.

Total population is normalized to 1. Each employed individual inelastically supplies one unit of labor. Aggregate labor supply is given by

$$N_t = \int_{a_{\min}}^{\infty} f\left(e, a_t\right) \mathrm{d}a_t.$$

Aggregate capital stock is

$$K_t = \sum_{s \in \{e,u\}} \int_{a_{\min}}^{\infty} a_t f\left(s, a_t\right) \mathrm{d}a_t,$$

where $f\left(s_t, a_t\right)$ is the probability density function over employment status and asset holdings.

Output is given by a constant-returns-to-scale production function which takes capital and labor as its inputs.

$$Y_t = F\left(K_t, N_t\right).$$

The factor prices are the marginal products of the two factors of production

$$r_t = F_1\left(K_t, N_t\right) \quad \text{and} \quad w_t = F_2\left(K_t, N_t\right).$$

Budgets are balanced period-by-period

$$\underbrace{\omega \int_{a_{\min}}^{\infty} f\left(u, a_t\right) da_t}_{\text{expenditures}} = \underbrace{\tau F\left(K_t, N_t\right)}_{\text{revenue}}.$$

Computational procedure

1. **Step 1** Compute the stationary employment N.

   In the model economy in question, employment N_t does not depend on the prices r_t and w_t or the distribution of assets a_t in period t. We can therefore simply iterate on the Markov transition matrix until it converges to the stationary employment.

   Step 2 Make initial guesses of the aggregate capital stock K and the tax rate τ

   Step 3 Compute the wage rate w and the interest rate r

Step 4 Compute the households' decision rules.

We have the same set of options for doing this as we have for solving the standard stochastic neoclassical growth model.

Step 5 Since the wealth distribution is continuous and, hence, is an infinite-dimensional object, it can only be computed approximately. At least three methods

(a) Compute an approximation for the distribution function on a discrete number of grid point over the assets

(b) Using Monte-Carlo simulations by constructing a sample of households and tracking them over time

(c) Assuming a specific functional form of the distribution function and use iterative methods to compute the approximation.

For the first approach, the idea is roughly that given the stationary decision rules the invariant probability density function must map into itself. The steps are

(a) Place a grid on the asset space

(b) Assume a uniform probability density function over all points in the asset state space.

(c) Compute the inverse of the decision rule $a'(s,a)$

(d) Iterate on

$$f_{i+1}(s',a') = \sum_{s\in\{e,u\}} \pi(s' \mid s) f_i\left(s, a'^{-1}(a',s)\right)$$

until f converges

Step 6 Compute taxes T and the capital stock

$$K = \sum_{s\in\{e,u\}} \int_{a_{\min}}^{\infty} af(s,a)\,da,$$

Step 7 Compute the tax rate τ that solves the government budget.

Step 8 Update K and τ and return to Step 2 if necessary.