ECON5160: Some solutions to exercises in Schweder (2009)

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S1 (Exercise 1)

With $g(x;a,b)$ the gamma density with shape $a>0$ and rate $b>0$, and $c>-\alpha$

$$\begin{aligned}
EX^c &= \int_0^\infty x^c g(x;\alpha,\beta)\,dx \\
&= \int_0^\infty x^c \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx \\
&= \frac{\beta^{\alpha+c}\Gamma(\alpha+c)}{\beta^\alpha\Gamma(\alpha)} \int_0^\infty g(x;\alpha+c,\beta)\,dx = \beta^c \frac{\Gamma(\alpha+c)}{\Gamma(\alpha)}.
\end{aligned}$$

Thus

$$\begin{aligned}
EX &= \alpha\beta \\
var(X) &= EX^2 - (EX)^2 \\
&= \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2 = \alpha\beta^2 \\
E(X-EX)^3 &= EX^3 - 3\left(EX^2\right)EX + 3(EX)(EX)^2 - (EX)^3 \\
&= \alpha(\alpha+1)(\alpha+2)\beta^3 - 3\alpha(\alpha+1)\beta^2\alpha\beta + 2(\alpha\beta)^3 \\
&= \alpha\beta^3\left[\left(\alpha^2+3\alpha+2\right) - 3\left(\alpha^2-\alpha\right) + 2\alpha^2\right] = 2\alpha\beta^3.
\end{aligned}$$

S4

The Poisson likelihood of an observation x is $L(\mu;x) = \frac{\mu^x}{x!}e^{-\mu}$, and the prior for μ is the gamma density $g(\mu;\alpha,\rho)$ where the rate $\rho = 1/\beta$. The posterior is

$$\begin{aligned}
f(\mu) &= \frac{L(\mu;x)g(\mu;\alpha,\rho)}{\int_0^\infty L(m;x)g(m;\alpha,\rho)\,dm} \\
&= \frac{\mu^{x+\alpha-1}e^{-\mu-\rho\mu}}{\int_0^\infty m^{x+\alpha-1}e^{-m(1+\rho)}dm} = g(\mu;x+\alpha,1+\rho).
\end{aligned}$$

The scale parameter of the posterior, which indeed is gamma, is $1/(1+\rho) = \beta/(1+\beta)$.

S7

As long as $\theta > 0$ C wins the election with certainty, while the chance is 1/2 each time $\theta = 0$. The number of successive wins for C when $\theta = 0$, N, has thus the mentioned geometric distribution, and since V_n is the number of wins from (and including) its nth win when $\theta = 0$ to its next return to $\theta = 0$ $T = \sum_{n=1}^{N} V_n$. $E(T) = EN \cdot EV = 2EV$. $EV = 1/\pi_0 = \frac{2r+q}{r-p}$. $2\frac{2r+q}{r-p}$ is also the expected number of successive wins for party L. Since $\pi_j^C = \pi_{-j}^L$

$$
\begin{aligned}
P\left(In \leq x|C\right) &= \sum_{j=0}^{\infty} \pi_j^C \Phi\left(\frac{\log(x) - \left(\mu + \frac{j}{a}\right)}{\tau}\right) \\
&= \sum_{j=0}^{\infty} \pi_{-j}^L \Phi\left(\frac{\log(x) - \left(\mu + \frac{j}{a}\right)}{\tau}\right) \\
&= \sum_{j=-\infty}^{0} \pi_j^L \Phi\left(\frac{\log(x) - \left(\mu - \frac{j}{a}\right)}{\tau}\right) = P\left(In \leq x|L\right).
\end{aligned}
$$

The income distribution is thus identical in the two situations. If each person has its y and $z = x - \theta$, such that its log income is $y + x$ under C and $y - x$ under L, those with negative z vote L, and have then log income $y - z - \theta$ while it had been $y + z + \theta$ under C. Two persons, one with the same absolute value of z but of opposite sign will thus have the same income pair in two situations of θ of same absolute size, but with opposite sign. They have just changed positions in the income distribution.

S8

Assume the three-state chain to be irreducible. Then it is either aperiodic, or it has period 3. Assume further that all eigenvalues of $\mathbf{P}$ are distinct, allowing our version of the Peron-Frobenius theorem to apply. In case the chain is aperiodic, let λ_2 and λ_3 be the two distinct eigenvalues of norm $|\lambda| < 1$. When both these are positive, as is the case for Lind's $\mathbf{P}$, we have the P-F matrix

$$
\begin{aligned}
P_h &= \mathbf{U}\mathbf{\Lambda}^h\mathbf{V}' \\
&= \mathbf{U}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 + h\log\left(\lambda_2\right) & 0 \\ 0 & 0 & 1 + h\log\left(\lambda_2\right) \end{bmatrix}\mathbf{V}' + o(h) \\
&= \mathbf{U}\mathbf{V}' + h\log\left(\lambda_2\right)u_2v_2' + h\log\left(\lambda_3\right)u_3v_3' + o(h) \\
&= \mathbf{I} + h\left(\log\left(\lambda_2\right)u_2v_2' + \log\left(\lambda_3\right)u_3v_3'\right) + o(h)
\end{aligned}
$$

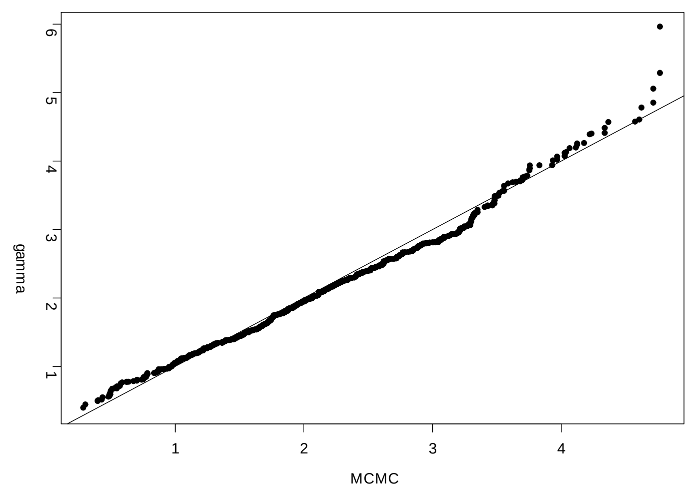
when h is small. For this to be a probability matrix, the matrix $\left(\log\left(\lambda_2\right)u_2v_2' + \log\left(\lambda_3\right)u_3v_3'\right)$ must have negative numbers on the diagonal, non-negative numbers off the diagonal, and it must have rows summing to 1. These conditions are satisfied for Lind's $\mathbf{P}$. It is also true that in his case $\mathbf{P}^{1/4}= \mathbf{U}\mathbf{\Lambda}^{\frac{h}{1}}\mathbf{V}'$

S11

The transition kernel for a random walk has the form $K(u,t) = k(t-u)$ where k is the density of the step $V = X_{n+1} - X_n$. The detailed balance equation implies for a function g, $\int k(t-u)\, g(u)du = \int k(t-u)\, g(t)du = g(t)$. That is $Eg(t-V) = g(t)$ for all t. This can only happen when g is constant, and for nontrivial solutions, g not equal to 0 everywhere, g does not have final integral.

S13

The posterior is gamma, as explained in Section 2, with shape parameter $\alpha + X = 6$ and scale parameter $\beta/(\beta+1) = 1/3$.The numerator in Bayes formula is proportional to $g(\lambda) = \lambda^5 e^{-3\lambda}$. When the proposal kernel simply is the standard normal random walk, with reflection at zero (to make the chain move on the positives) making $q(u,t) = \varphi(t-u) + \varphi(t+u)$ for $t > 0$ we get $q(u,t) = q(t,u)$ and $\rho(t,u) = \min\left\{\frac{g(u)q(u,t)}{g(t)q(t,u)}, 1\right\} = \min\left\{\left(\frac{u}{t}\right)^5 e^{3(t-u)}, 1\right\}$. I have simulated the chain in 1000 steps. The qq-plot versus the correct gamma distribution is shown in Figure .



Exerise 13: qq-plot for 1000 MCMC simulations based on a normal random walk reflected at the origin, against the gamma(shape=6, scale=1/3) distribution. The straight line is the diagonal.