ECON5160: Solutions to some problems in Taylor and Karlin (1998)

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January 2009

TK IP2.13 (problem 13, section 2, part I in Taylor & Karlin 1998)

$(U,V) \leftrightarrow (X,Y)$ is neither 1 to 1 or differentiable. The rule in 2.6 does therefore not apply directly. But $U \leq u \& V > v \Leftrightarrow v < X \leq u \& v < Y \leq u$ and has probability $(u-v)^2$. Since then the joint cumulative distribution function is $F_{UV}(u,v) = P(U \leq u \& V \leq v) = P(U \leq u) - P(U \leq u \& V > v) = u^2 - (u-v)^2$,

$$f_{UV}(u,v) = \frac{d^2}{dudv} F_{UV}(u,v) = \begin{matrix} 2 & 0 < u < v < 1 \\ 0 & \text{otherwise.} \end{matrix}$$

TK IP3.4

$$\begin{aligned} E\frac{1}{1+U} &= \sum_{u=0}^{\infty} \frac{1}{1+u} \frac{\mu^u}{u!} e^{-\mu} \\ &= \frac{1}{\mu} \sum_{u=0}^{\infty} \frac{\mu^{u+1}}{(1+u)!} e^{-\mu} \\ &= \frac{1}{\mu}\left(1 - e^{-\mu}\right) \end{aligned}$$

TK IP3.11

For $w = v - u > 0$ and $u \geq 0$.

$$\begin{aligned} P(U = u \& W = w) &= P(U = u \& V = v) \\ &= P(X = u \& Y = v) + P(X = v \& Y = u) \\ &= 2\pi^u (1-\pi) \pi^v (1-\pi) \\ &= 2\pi^{2u} \pi^w (1-\pi)^2 \\ &= 2\pi^w \frac{1-\pi}{1+\pi} \cdot . \left(1-\pi^2\right) \left(\pi^2\right)^u = P(W = w) P(U = u), \end{aligned}$$

and for $w = v - u = 0$ and $u \geq 0$

$$\begin{aligned} P(U = u \& W = 0) &= \left(\pi^u (1-\pi)\right)^2 \\ &= \frac{1-\pi}{1+\pi} \cdot \cdot \left(1-\pi^2\right)\left(\pi^2\right)^u = P(W=0)\, P(U=u). \end{aligned}$$

TK IP4.3

$W = X - Y = \left(X - \theta + \frac{1}{2}\right) - \left(Y - \theta + \frac{1}{2}\right) = U_1 - U_2$ where U_1, U_2 are independent and uniformly distributed on (0, 1), not dependent on θ. The density is

$$\begin{aligned} f_W(w) &= \int_0^1 f_U(u)\, f_U(w+u)\, du \\ &= \begin{matrix} \int_{-w}^1 du = 1 + w & -1 \leq w < 0 \\ \int_w^1 du = 1 - w & 0 \leq w \leq 1 \\ 0 & |w| > 1 \end{matrix} \end{aligned}$$

TK IP4.5

$$\begin{aligned} P(X < Y) &= \int_0^\infty 2e^{-2x} \left(\int_x^\infty 3e^{-3y} dy \right) dx \\ &= \int_0^\infty 2e^{-2x} e^{-3x} \left(\int_x^\infty 3e^{-3(y-x)} dy \right) dx \\ &= \frac{2}{5} \int_0^\infty 5e^{-5x} dx = \frac{2}{5}. \end{aligned}$$

TK IP5.7

$$\begin{aligned} P(V > v) &= P(\cap X_i > v) \\ &= \prod P(X_i > v) \\ &= \prod e^{\lambda_i v} = e^{(\sum \lambda_i) v}. \end{aligned}$$

V is exponentially distributed with rate parameter $\sum \lambda_i$.

TK IP5.8

a)

$$P(V_n > v) = P(\cap X_i > v) = (1 - v)^n$$

b)

$$\begin{aligned} P(W_n > w) &= P(V_n > w/n) \\ &= \left(1 - \frac{w}{n}\right)^n \\ &= \left(\left(1 - \frac{w}{n}\right)^{-n/w} \right)^{-w} \to e^{-w}. \end{aligned}$$

V_n is asymptotically exponentially distributed, and the rate of convergence is n^{-1}, i.e. $nV_n \xrightarrow{D}$standard exponential, rather than $1/\sqrt{n}$ which we have in the Central limit theorem.

TK IIP1.9

Since $P(X = x|N = n) = 1/(n+2) \quad n = x-1, x, \cdots,$

$$\begin{aligned} f_X(x) &= \sum_{n=x-1}^{\infty} \frac{1}{n+2} \frac{1^n}{n!} e^{-1} \\ &= \sum_{n=x-1}^{\infty} \frac{n+1}{(n+2)!} e^{-1} \\ &= \sum_{n=x-1}^{\infty} \left(\frac{1}{(n+1)!} - \frac{1}{(n+2)!} \right) e^{-1} \\ &= \sum_{m=x}^{\infty} \frac{1}{m!} e^{-1} - \sum_{m=x+1}^{\infty} \frac{1}{m!} e^{-1} = \frac{1}{x!} e^{-1}. \end{aligned}$$

TK IIP2.1

a)

$$\begin{aligned} M &= EE[X_N|X_1] \\ &= \int_A^{\infty} xf(x)dx + E[X_N|X_1 < A] P(X_1 < A) \\ &= \int_A^{\infty} xdF(x) + (1-\alpha) M. \end{aligned}$$

b) This determines M for $0 < \alpha$.

$$M = \int_A^{\infty} xdF(x)/\alpha = E[X|X > A].$$

c) The memoryless property of the exponential distribution is $X|X > A \overset{D}{=} A + X$, and consequently $E[X|X > A] = A + EX = A + 1/\lambda$

d)

$$\begin{aligned} M &= \int_A^{\infty} xdF(x)/\alpha \\ &= \int_A^{\infty} x\lambda e^{-\lambda x} dx \cdot \frac{1}{e^{-\lambda A}} \\ &= \int_A^{\infty} (x - A + A) \lambda e^{-\lambda(x-A)} dx \\ &= \int_0^{\infty} x\lambda e^{-\lambda x} dx + A \int_0^{\infty} \lambda e^{-\lambda x} dx = 1/\lambda + A. \end{aligned}$$

TK IIP3.1

With $B_1, B_2, \cdots$ independent Bernoulli variables,

a)

$$\begin{aligned} Z &= \sum_{i=1}^{N} B_i \\ EZ &= EB_i \cdot EN = p\lambda \\ varZ &= Evar\left[Z|N\right] + varE\left[Z|N\right] \\ &= E\left(Np\left(1-p\right)\right) + var\left(Np\right) \\ &= \lambda p\left(1-p\right) + \lambda p^2 = \lambda p \end{aligned}$$

b)

$$\begin{aligned} f(z) &= \sum_{n=z}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} \binom{n}{z} p^z \left(1-p\right)^{n-z} \\ &= \frac{\left(\lambda p\right)^z}{z!} e^{-\lambda} \sum_{n=z}^{\infty} \frac{1}{(n-z)!} \left(\lambda\left(1-p\right)\right)^{n-z} \\ &= \frac{\left(\lambda p\right)^z}{z!} e^{-\lambda} e^{-\lambda(1-p)} = \frac{\left(\lambda p\right)^z}{z!} e^{-\lambda p} \end{aligned}$$

Poisson λp!

TK IIP3.4

a)

$$\begin{aligned} ES_N &= \lambda\mu \\ varS_N &= Evar\left[S_N|N\right] + varE\left[S_N|N\right] \\ &= \lambda\left(\sigma^2 + \mu^2\right) = v^2. \end{aligned}$$

b) $EN = \lambda$, $varN = \lambda/p$

$$\begin{aligned} ES_N &= \lambda\mu \\ varS_N &= Evar\left[S_N|N\right] + varE\left[S_N|N\right] \\ &= \lambda\sigma^2 + \frac{\lambda}{p}\mu^2 = w^2. \end{aligned}$$

c)

$$\frac{w^2 - v^2}{\lambda\mu^2} = \lambda \to \infty$$

TK IIP4.2

X is Poisson λp (TKIIP3.1). For the same reason, Y is Poisson $\lambda(1-p)$.

$$\begin{aligned}
P(X=x\cap Y=y) &= P(X=x\cap N=x+y) \\
&= P(X=x|N=x+y)\,P(N=x+y) \\
&= \binom{x+y}{x} p^x (1-p)^y \frac{\lambda^{x+y}}{(x+y)!} e^{-\lambda} \\
&= \frac{(\lambda p)^x}{x!} e^{-\lambda p} \frac{(\lambda(1-p))^y}{y!} e^{-\lambda(1-p)} = P(X=x)\,P(Y=y).
\end{aligned}$$

Independent!

TK IIP4.6

$$\begin{aligned}
P(N>n) &= P\left(\cap_{i=1}^n X_i \le X_0\right) \\
&= \int_{-\infty}^{\infty} P\left(\cap_{i=1}^n X_i \le x\right) f(x)dx \\
&= \int_{-\infty}^{\infty} F(x)^n f(x)dx \\
&= \int_0^1 u^n du = \frac{1}{n+1} \\
P(N=n) &= P(N>n-1) - P(N>n) \\
&= \frac{1}{n(n+1)} \qquad n=1,2,\cdots \\
EN &= \sum_{n=0}^{\infty} P(N>n) = \infty
\end{aligned}$$

TK IIP5.3

$$\begin{aligned}
E[X_{n+1}|X_1,X_2,\cdots,X_n] &= E[X_{n+1}|S_n] \\
&= E\left[2^{n+1}e^{-S_n-\varepsilon_{n+1}}|S_n\right] \\
&= 2^n e^{-S_n} 2E\left[e^{-\varepsilon_{n+1}}\right] \\
&= X_n 2E\left[1-F_\varepsilon(\varepsilon_{n+1})\right] = X_n
\end{aligned}$$

since $1-F_\varepsilon(\varepsilon_{n+1})$ is uniformly distributed over (0,1), and has mean 1/2.

TK IIP5.5

The process $\{X_n\}$ is a random walk on the positive integers, but with 0 as an absorbing state.

a) If $X_n=0$, $X_{n+1}=0$ and $E[X_{n+1}|X_1,X_2,\cdots,X_n]=0=X_n$. If $X_n=x>0$

$$\begin{aligned}
E[X_{n+1}|X_1,X_2,\cdots,X_n] &= E[X_{n+1}|X_n=x] \\
&= \frac{1}{2}(x+1)+\frac{1}{2}(x-1) = X_n.
\end{aligned}$$

b) By the maximal inequality, for all $\varepsilon > 0$

$$P\left(\cup_{j=1}^{\infty} X_j > N - \varepsilon | X_0 = i\right) \leq \frac{E\left[X_0 | X_0 = i\right]}{N - \varepsilon} = \frac{i}{N - \varepsilon}.$$

Consequently since N is an integer

$$P\left(\cup_{j=1}^{\infty} X_j \geq N | X_0 = i\right) \leq \frac{i}{N}.$$

With a strategy of quitting the game when the fortune of the player reaches $X_n = N$, the probability of quitting with a fortune of N is at most i/N. In part III it is argued that the game will end sooner or later with probability 1, and the probability of ending up broke is thus at least $1 - i/N$.