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ECON5160: Supplementary information on TK/PK ch. IV

1 A few bits and pieces

- Problem IV 3.2 says that a finite state aperiodic irreducible Markov chain is regular and recurrent. This result is curriculum.
- Notice that the definition given for aperiodicity ($\mathsf{Pr}[X_n = i | X_0 = i] > 0$ for all n large enough) differs from the book. To prove it is an equivalent definition, see TK p. 239 (PK p. 197) property 2, and for the converse implication: pick N and $N+1$ and apply the book's definition of periodicity.
- If the state space is finite, there has to be some recurrent state (and this is necessarily positive recurrent, as null recurrence is only possible for infinite state space).
- For a positive recurrent irreducible Markov chain, there is a *unique* stationary distribution.
- I introduced the matrix $\mathbf{\Pi}$ with all rows equal (to, say, $\boldsymbol{\pi}$). Notice that this can be written as $\mathbf{1}\boldsymbol{\pi}$ where $\mathbf{1}$ is a *column* vector of only ones.

The next part of this note will give a walk-through on what happens if the Markov chain is reducible. Since it fits neatly on a double page, I put a pagebreak here.

2 Reducible Markov chains

We will give an example with a process X living on a finite state space decomposable into three communicating classes, a generalization of the example given in TK p. 262 (PK p. 218) ff., where

- each class is a «communication» equivalence class,
- classes C_1 and C_3 are recurrent but will each trap the process, and
- class C_2 is transient.

Then the transition matrix $\mathbf{P}$ for X can be written as

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \mathbf{0} & \mathbf{E} \end{bmatrix} \begin{matrix} \} \text{ class } C_1 \\ \} \text{ class } C_2 \\ \} \text{ class } C_3 \end{matrix} \tag{1}$$

where all the boldfaced symbols are matrices. Then, in ten steps:

I) The representation (1) is assumed chosen so that the matrix dimensions match – e.g. $\mathbf{A}$, $\mathbf{B}$ and the lower-left $\mathbf{0}$ have the same width (= number of columns).

II) We assume that $\mathbf{A}$ and $\mathbf{E}$ have no null-columns (any such can be incorporated in the «$\mathbf{0}$»). Of course they have no null-rows either.

III) For class C_1 to trap the process, $\mathbf{A}$ (now assumed to contain no null-columns!) must be at least as high as wide; if not, it means that some state outside C_1 is accessible from some state in C_1.
Same argument goes for $\mathbf{E}$.

IV) For class C_1 to be an equivalence class (in the sense of communication), then under the «trap» assumption it must be at least as wide as high – if not, there is some state i in C_1 inaccessible from the other states in C_1, since by the trap assumption, X cannot reach i by exiting C_1.
Same argument goes for $\mathbf{E}$.

V) For this three-class case we may therefore assume $\mathbf{A}$ and $\mathbf{E}$ to be square. Then $\mathbf{C}$ is also square, and describes the one-step transition probabilities from states in C_2 to states in C_2.

VI) Observe that the matrix powers $\mathbf{P}^n$ of $\mathbf{P}$ also has the structure given in (1), with zeroes as indicated:

$$\mathbf{P}^n = \begin{bmatrix} \mathbf{A}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_n & \mathbf{C}_n & \mathbf{D}_n \\ \mathbf{0} & \mathbf{0} & \mathbf{E}_n \end{bmatrix} \begin{matrix} \} \ C_1 \\ \} \ C_2 \\ \} \ C_3 \end{matrix} \tag{2}$$

where $\mathbf{A}_n = \mathbf{A}^n$ (matrix power), $\mathbf{C}_n = \mathbf{C}^n$ and $\mathbf{E}_n = \mathbf{E}^n$ (but not similarly for $\mathbf{B}_n$ and $\mathbf{D}_n$).

VII) From the first time hitting C_1, the Markov property ensures that X will have the same probability distributions as a process Y which has $\mathbf{A}$ as transition matrix. Similar goes for $\mathbf{E}$ as the transition matrix for a process Z (from the time X hits C_3). Then the properties of Y and Z may be studied standalone, and the main issue concerning the reducibility property is «how to deal with transient states» (i.e. C_2).

VIII) Let T be the first exit time from C_2. Let $\mathbf{U}$ be the matrix of probabilities

$$u_{ij} = \mathsf{Pr}[X_T = j | X_0 = i] \quad \text{for } i \in C_2,\ j \in C_1,$$

and let similarly $\mathbf{V}$ have entries indexed over $j \in C_3$

$$v_{ij} = \mathsf{Pr}[X_T = j | X_0 = i] \quad \text{for } i \in C_2,\ j \in C_3.$$

By first-step analysis,

$$\begin{aligned} \mathbf{U} &= \mathbf{B} + \mathbf{CU} \qquad \text{so that} \qquad \mathbf{U} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{B} \\ \mathbf{V} &= \mathbf{D} + \mathbf{CV} \qquad \text{so that} \qquad \mathbf{V} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{D} \end{aligned}$$

(the transience of C_2 will grant invertibility.) If we merely want the probabilities u_i that X will hit (somewhere in) C_1, that will be the «sum over j for each i», namely

$$\mathbf{u} = \mathbf{U1} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{B1} \qquad \text{and similarly} \qquad \mathbf{v} = \mathbf{V1} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{D1},$$

where $\mathbf{1}$ is the column vector of ones.[1]

IX) By recurrence, we can find unique stationary distributions $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{A}$ for Y and $\boldsymbol{\zeta} = \boldsymbol{\zeta}\mathbf{E}$ for Z. Then the time averages (i.e. mean occupation ratios) converge, and to[2]

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m<n} \mathbf{P}^m = \begin{bmatrix} \mathbf{1}\boldsymbol{\pi} & \mathbf{0} & \mathbf{0} \\ \mathbf{u}\boldsymbol{\pi} & \mathbf{0} & \mathbf{v}\boldsymbol{\zeta} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}\boldsymbol{\zeta} \end{bmatrix}. \tag{3}$$

X) The limiting distributions may or may not exist, depending on the convergence or not of $\lim_n \mathbf{A}^n$ and $\lim_n \mathbf{E}^n$. If they converge, the corresponding entry is given as in (3). In the TK example, $\mathbf{A}$ is regular so C_1 (i.e. Y) is aperiodic and $\boldsymbol{\pi}$ is a limiting distribution for Y, but C_3 is periodic. Then $\lim_n \mathbf{A}^n$ converges and its limit $\boldsymbol{\Pi}$ is $= \mathbf{1}\boldsymbol{\pi}$, and furthermore $\mathbf{u}\boldsymbol{\pi} = \mathbf{U1}\boldsymbol{\pi} = \mathbf{U}\boldsymbol{\Pi}$. However, $\lim_n \mathbf{E}^n$ diverges. Then formula in TK top of p. 263 (PK middle p. 219) corresponds to

$$\mathbf{P}^n \text{ «tends to» } \begin{bmatrix} \mathbf{1}\boldsymbol{\pi} & \mathbf{0} & \mathbf{0} \\ \mathbf{u}\boldsymbol{\pi} & \mathbf{0} & \star \\ \mathbf{0} & \mathbf{0} & \star \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Pi} & \mathbf{0} & \mathbf{0} \\ \mathbf{U}\boldsymbol{\Pi} & \mathbf{0} & \star \\ \mathbf{0} & \mathbf{0} & \star \end{bmatrix} \quad \text{(«}\star\text{» for divergent submatrices)} \tag{4}$$

[1] The corresponding relation on TK p. 262 (PK p. 219) are $u_2 = \frac{1}{3}(1) + 0(1) + \ldots$ and $u_3 = \frac{1}{6}(1) + \frac{1}{6}(1) + \ldots$ – observe from the next line therein that the constant coefficient is the sum of the «(1)» coefficients for each row.

[2] The corresponding formula in TK is middle p. 263 (PK bottom p. 219). Note that the «center» matrix is null since transience of C_2 implies $\lim_n \mathbf{C}^n = \mathbf{0}$.