ECON 5160 Apr 29

* A bit on portfolio choice:
  the portfolio separation theorem

* Arbitrage-free derivatives pricing

Both: we assume frictionless markets

I

Portfolio choice

* This course is not about dynamic optimization
  → But: the special cases to be treated today, reduce to <u>static</u> Lagrange problems

* The <u>portfolio separation</u> (a.k.a. "<u>mutual fund</u>") theorem in a nutshell:
  For some distributions / preferences, there is no welfare loss in restricting investment to some (a <u>few</u>) "funds", i.e. indexes
  → Prototypical case: <u>two</u>-fund separation: reduces the problem to (one-dimensional) allocation between these two vectors
  → two-fund <u>monetary</u> sep.: if the "bank" can be taken to be one of the two funds.

Various versions of the portfolio separation property

→ Preferences: some utility functions (incl power/log, exp) ... not this course.

→ Distributions: What probability dist. on the returns vector, are such that k funds will keep all expected utility maximizers in optimum?

Models like CAPM assume a mean—variance trade-off, which leads to 2fund sep. — monetary if there is money (otherwise, "zero-beta CAPM").

→ Usual mean—variance approach: min variance for given mean
→ assumes risk aversion

→ more general: max mean for given variance.

A single period model:

→ Investment opportunities ("stocks" — but no limited liability assumed!) $S_1, \ldots, S_n$ ("risky") and one "safe" opportunity S_0.

→ shall cover the case when no "safe" opportunity exists by forbidding investment in S_0

→ Market ~~value~~ return on S_0: $r\, S_0$

on S_i, $i>0$: $(r_i + \sigma_i \cdot Z)\, S_i$

random vector with $EZ = 0$

→ Invest: have ν_i units of S_i, $i>0$; then in S_0: $\nu_0 = \left(x - \sum_{i>0} \nu_i S_i\right) / S_0$ units.

→ Return on portfolio:

$$\sum_{i>0} (r_i + \sigma_i \cdot Z)\, \nu_i S_i + rx - \sum_{i>0} r\, \nu_i S_i$$

Put $u_i = \nu_i S_i$, $\mu_i = r_i - r$ ("excess return")

Forget about "rx" (portfolio independent).

Consider:

$$u^T (\mu + \sigma Z)$$

vectors matrix random vector with $EZ = 0$

$*$ $u^T(\mu + \sigma Z)$, $EZ = 0$

→ No arbitrage:

- those u for which $u^T\sigma = 0$ also have $u^T\mu = 0$
- If such a $u \neq 0$ exists, then there is a <u>redundant</u> investment opportunity – remove from model and reformulate!

⇝ assume: $M = \sigma\sigma^T$ positive definite.

* $u^T(\mu + \sigma Z)$, Z multinormal, $EZ = 0$:

→ If Z is multinormal, then $u^T(\mu + \sigma Z)$ is univariate

$$\mathcal{N}(u^T\mu, \underbrace{u^T \sigma \sigma^T u}_{u^T M u})$$

→ Fix u. Assume there is a u^* so that $(u^*)^T\mu > u^T\mu$

$$(u^*)^T M u^* = u^T M u$$

Then $\text{Return}^{u^*} \stackrel{\text{distr.}}{=} \text{Return}^{u} + \underbrace{(u^* - u)^T\mu}_{>0}$

so Return^{u^*} <u>first-order stochastic dominates</u> Return^{u}.

⇒ <u>all</u> expected utility maximizers (with increasing utility function, but no concavity assumed), prefer (possibly weakly) Return^{u^*} to Return^{u}.

So: no-one will choose u^* unless it solves

$$\max u^T \mu \quad \text{subject to } u^T M u = Q^2$$

$\uparrow$ $\geq 0 \ldots$

* Two-fund (monetary) separation (1):

$$u^* = \underbrace{\frac{Q}{\sqrt{\mu^T (M^{-1})^T \mu}}}_{\text{scalar}} \quad \underbrace{(M^{-1}\mu)}_{\text{vector, "fund"}}$$

Note: the choice of Q will be individual

Proof: Lagrange.

* Two-fund (monetary) separation (2)

Same holds if u is restricted to a cone K for example if short-sale is disallowed for some opportunities:

Proof:

If f solves

$$\max_u u^T \mu \quad \text{s.t } u^T M u = 1, \ u \in K$$

then $u^* = Qf$ solves

$$\max_u u^T \mu \quad \text{s.t. } u^T M u = Q^2, \ u \in K.$$

Two-fund separation in the absence of money/"safe" opportunity:

Additional constraint: $u^T \mathbb{1} = x$

$\mathbb{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ wealth

$$\max_u u^T \mu \quad \text{s.t.} \quad u^T M u = Q^2, \quad u^T \mathbb{1} = x$$

$$L = u^T \mu - \lambda_0 (u^T M u - Q^2) - \lambda_1 (u^T \mathbb{1} - x)$$

F.o.c:

$$2\lambda_0 M u = \mu - \lambda_1 \mathbb{1}$$

$$u = \underbrace{M^{-1} \mu}_{\text{fund (as before!)}} \cdot \frac{1}{2\lambda_0} - \underbrace{M^{-1} \mathbb{1}}_{\text{fund}} \cdot \frac{\lambda_1}{2\lambda_0} \quad \text{(no bank)}$$

Some fiddling around with

→ $\lambda_0 = 0$?

→ constraint qualification?

Note: Can <u>not</u> add a cone constraint!

* Elliptical distributions
(a.k.a: "elliptically contoured" distributions)

This construction works also, e.g, if Z is a multinormal variable <u>times</u> an independent positive r.v. R
(Proof: if $u^{*T} \sigma Z \stackrel{d}{=} u^T \sigma Z$ then $R u^{*T} \sigma Z \stackrel{d}{=} R u^T \sigma Z$)

Furthermore, it works if $Z \in \mathbb{R}^m$ is uniform on the $\mathbb{R}^{m-1}$ - <u>unit sphere</u>.
(Sketch for $m = 3$, $\sigma = I$:
$u^T Z$ depends only on $u^T u$, since Z is rotationally independent

)

* Elliptical distributions

Def 1: $E[\exp i \vartheta^T Z]$ is a function of $\vartheta^T M \vartheta$, some positive-def. M.

Def 2: start with uniform on unit sphere, multiply by a σ s.t. $M = \sigma \sigma^T$ is pos. def, multiply by independent $R \geq 0$, move by adding a μ.

* Fact:
Elliptical distributions are precisely those for which <u>regression is linear</u>.

* "Mean" and "variance" may even be infinite if $E[R] = \infty$.

But R does not matter for the two-fund separation result

(However 1: risk averse agents will choose $u = 0$ if ....

2: the expectation in "expected utility" may even diverge, so we have to resort to the stoch. dominance assumption

* Multiple periods with intermediate consumption:

$$u(t)^T\left(\mu + \sigma Z(t)\right) - C(t) \quad , t=0,\dots$$

each $Z(t)$ drawn indep.

→ Assume you consider the strategy $(u(t), C(t))$. Define $(u^*(t), C^*(t))$ as follows:

u^* mean-variance efficient with $(u^*)^T M u^* = u^T M u$

$\Rightarrow$ then $(u^*)^T \mu \geq u^T \mu$

$$C^* = C + \underbrace{(u^* - u)^T \mu}_{\geq 0}$$

Wealth processes identical in law, but $(u^* - u)^T \mu$ (≥ 0) more to consume!

(We can have $\mu = \mu(t)$ and $\sigma = \sigma(t)$ too, even stochastic if independent of Z; then the fund(s) change in time, but are the same for all agents!)

* Continuous time with intermediate consumption:
  - → $C(t)$ nondecreasing, i.e. $dC \geq 0$
  - → replace "Z" by "dZ" with independent increments (*)
  - → A <u>self-financing</u> portfolio construction leads to
    $$dX = u^T(\mu\, dt + \sigma\, dZ)$$
    if the S_i have dynamics
    $$dS_i = S_i(\mu_i\, dt + \sigma_i\, dZ)$$
  - → Financing consumption:
    $$dX = u^T(\mu\, dt + \sigma\, dZ) - dC$$
  - → Given (u, C): construct
    $$u^* \text{ s.t. } (u^*)^T M u^* = u^T M u$$
    $$(u^*)^T \mu \geq u^T \mu$$
    $$C^* = C + \int_0^t \underbrace{(u^*(s) - u(s))^T \mu(s)}_{\geq 0}\, ds$$

(*) Only possible for the so-called <u>infinitely divisible</u> distributions. Hence, actually, the discrete-time result is more general!

II

* Arbitrage-free pricing

Ex: A stock has price 100 today. A European call option maturing in one year with strike 105, is:

"option" : right, but not obligation

"call" : ... to buy

"European": ... at maturity, no sooner, no later...

"strike" : for 105.

Q: What is the price of this option?

First answer:

What is the market price of a banana? Supply & demand decide!

But: Price cannot be as much as the stock price (assuming limited liability): If it were, you could

→ sell the option and buy the stock (and possibly have money left!)

→ hold to maturity

→ at maturity, either sell to the option holder (if option exercised) or at the market – at worst, you get 0 for this.

Bottom line: there must be some reasonable relationship between derivative & underlying

* Arrow-Debreu security:
Pays 1 if a particular event occurs,
0 otherwise.

* Prototypical example: stock starts at 1, and goes to

g , "good" state of the world
b , "bad" —— " ——

where $g > b$. We assume we can borrow at certain rate, $1+r \in (b, g)$ (WHY?)

Option with strike $k \in (b, g)$ (other cases uninteresting)

is $(g-k)$ Arrow-Debreu securities on the event "g".

→ Consider the following strategy:
Initially, write (= sell) the option at market price c. Buy the particular worth of $\xi = \frac{g-k}{g-b}$ (<1) of the stock.
You will have to borrow $(\xi - c)$.
At maturity, you own:

$$\xi S - (S-k)_+ - (\xi - c)(1+r)$$

$$= \underbrace{\frac{g-k}{g-b}(S-b) - (S-k)_+}_{=0 \text{ regardless of } S=g \text{ or } S=b!} + \xi b - (\xi - c)(1+r)$$

→ So : Must have $\xi b = (\xi - c)(1+r)$

$$c = \frac{g-k}{g-b}\left(1 - \frac{b}{1+r}\right)$$

* If the call were priced <u>higher</u> than this, then one could
  - → start with zero wealth
  - → perform the trades as described
  - → end up with something >0 <u>with 100% probability</u>.

* If the call were priced lower:
  - → do the <u>reverse</u> transaction.

* Observe: the <u>probabilities</u> of g, resp. of b, do not enter the price.

(* .... or do they? Will return to that)

* This price is not the expected discounted payoff unless the probabilities are as follows: Put $q = \Pr[g]$. Then

$$E\left[\frac{1}{1+r}(S-k)_+\right] = \frac{q}{1+r}(g-k) \quad \text{while}$$

$$c = \left(1-\frac{b}{1+r}\right)\frac{(g-k)}{g-b}$$

which are equal iff $q = \dfrac{1+r-b}{g-b}$ (obs.: this is <u>always</u> $\in (0,1)$ by assumption).

But if $Pr[g] = \frac{1+r-b}{g-b}$, then

expected discounted <u>stock value</u>

$E\left[\frac{1}{1+r} S\right]$, equals

$$\frac{1}{1+r}\left[\frac{1+r-b}{g-b}\, g + \left(\frac{g-b}{g-b} - \frac{1+r-b}{g-b}\right) b\right]$$

$= \ldots = 1$ = <u>today's price</u> !

SO: the probability measure $Q = \begin{array}{l} Pr_Q[g] = q \\ Pr_Q[b] = 1-q \end{array}$

for which

$$E_Q\left[\frac{1}{1+r} S\right] = \text{today's stock price}$$

also satisfies

$$E_Q\left[\frac{1}{1+r} \text{ option payoff}\right]$$

$$= \text{today's option price.}$$

Q is called the "risk neutral measure" or
the "equivalent martingale measure".

→ after discounting, → the stock is a Q-martingale

→ the option —— " ——

So: * the actual probability measure decides today's price (together with the market's attitude towards risk!)

* the "risk neutral measure" which would yield the same price in a risk neutral market, also prices <u>any</u> derivative.

Q: have we used any other properties of the "physical" probability measure?

A: Yes. Remember we assumed that only two states, "g" and "b", had positive probability.
The model assumes <u>null sets</u> known!

Q: What about the assumption
$g > 1+r > b$?

A: Assume both g, b were $> 1+r$.
Then the market would – <u>today</u> – value the stock as > 1. And so if $g > b \geq 1+r$
Similar: < 1 if g, b both < 1.
And also so if $1+r \geq g > b$.

Q: Is the <u>two-state</u> (g, b) assumption crucial?

A: Partly. With more than two states, there must still be:

→ a measure Q such that discounted expected value = today's.

IOW: All prices are Q-martingales. ("The fundamental theorem")

<u>But</u>: • Q is no more uniquely given from the null sets.

• There will be a range of possible pricing measures. The market will pick one.

Q: What about multiple periods?

A: Fix the # of periods T. Do the procedure recursively for $t = T-1, \ldots, 0$, rebalancing each step.

(If you cannot trade, "merge" two or more periods.)

Bank: e^{rt}

Q: What about continuous time?

→ Shall only consider geometric SDE prices:

$$dS = S(t^-)\left(\mu\,dt + \sigma\,dB + \gamma\,d\tilde{N}\right)$$

$= N-\lambda t$, martingale

→ ... and only (for now) "Markov" ₮-claims $H(S(T))$ (not path-dependent claims),

* Case $\gamma = 0$ (G.B.M.):

→ "Delta-hedging": hold the derivative, shortsell $\frac{\partial V}{\partial S}$

- Price at time t: $V(t, S(t))$
- Value of stock holdings: $S\frac{\partial V}{\partial S}$, which by self-financingness evolves as $\frac{\partial V}{\partial S}\,dS$.
- Value dynamics:

$$\left(V_t' + \tfrac{1}{2}\sigma^2 S^2 V_{SS}''\right)dt + \underbrace{V_S'\,dZ - V_S'\,dS}_{=0}$$

... must grow @ riskless rate r, otherwise arbitrage.

Value: $V - SV_S'$, so

$$\underbrace{r\left(V - SV_S'\right) = V_t' + \tfrac{1}{2}\sigma^2 S^2 V_{SS}''}_{\text{The Black–Scholes PDE.}}$$

- This PDE with the boundary condition

$$\lim_{t\to T} V(t,x) = H(x), \quad \text{all } x$$

gives arbitrage-free price.

* The "riskneutral pricing" side of the story

→ Turns out: again, for some probability measure Q, we have $V(0,x) = E_Q[e^{-rT} H(S^x(T))]$

where "S^x" means "S with $S(0)=x$"

→ When $S = x\exp((\mu - \frac{1}{2}\sigma^2)t + \sigma B(t))$ (gBm),

$E_Q[Y] = E[Y\,G(T)]$ where G is the martingale $G(t) = \exp(\theta B(t) - \frac{\theta^2}{2}t)$, $\theta = \frac{\mu - r}{\sigma}$ ("the market price of risk")

→ To see this, calculate by Itô's formula

$$E[e^{-rT} G(T) V(T, S^x(T))] - \underbrace{e^{-r\cdot 0} G(0)}_{1} V(0,x)$$

$$= E\int_0^T e^{-rt} G(t) \{\underbrace{-rV + V_t' + rSV_s' + \tfrac{1}{2}\sigma^2 S^2 V_{ss}''}_{=0,\ \text{B-S PDE}}\} dt$$

$$+ E\int_0^T \text{something } dB$$

(Actually, $e^{-rT} G(T) V(T, S^x(T))$ is a martingale)

So, $V(0,x) = E[e^{-rT} G(T) H(S^x(T))]$

holds regardless of μ. Putting $\mu = r$, then

$$V(0,x) = E[e^{-rT} \cdot 1 \cdot H(Z^x(T))]$$

Where $dZ = Z(r\,dt + \sigma dB)$, $Z(0) = x$

r, not μ!

(Can we really "put $\mu = r$"? Well, that would be the situation of a risk neutral market, which the formula is valid for!)

This gives a cookbook for Black–Scholes prices:

→ replace drift μ by risk-free r

→ discount

→ take (ordinary) expectation.

The call formula: Payoff $(S(T)-K)_+$ at T.

Price (@ 0): $E_Q\left[e^{-rT}(S(T)-K)_+\right]$

$$= E\left[e^{-rT}(Z(T)-K)_+\right]$$

which turns out to equal

$$\mathcal{N}(d_+)\,S - N(d_-)\,\underbrace{Ke^{-rT}}_{\text{P.V. of strike}}$$

spot price, i.e. P.V. of stock

where
$$d_\pm = \frac{\ln(S/Ke^{-rT}) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

* "Cookbook" also works for path-dependent derivatives!

* Extensions: e.g., if we do not have a constant interest rate r, but an independent zero-coupon bond paying 1 @ time T, then replace "Ke^{-rT}" by the bond's spot price.

* Completeness:
  - → The cases of unique Q (and unique price): hedging argument. Complete markets.
  - → If not: incomplete.
  - → Rule: Market complete (and unique prices) if you have just as many tradeables as noise sources
    - → jump + Brownian, one stock: incomplete
    - → jumps of different amplitude: — " —
    - → two Brownians? Depends!
      - → if you can read off each separately, e.g. issue a T-claim $F_1(B_1(T))$ and one $F_2(B_2(T))$, then one stock is insufficient.
      - → But if you only issue claims on S (path-dependent OK!) then put $\sigma_1 B_1 + \sigma_2 B_2 = \sigma B$ (where $\sigma = (\sigma_1^2 + \sigma_2^2)^{1/2}$). Complete!
    - → two jump processes with same amplitude? As two Brownians:
      $$\gamma\, dN_1 + \gamma\, dN_2 = \gamma\, dN$$
      with $\lambda = \lambda_1 + \lambda_2$.

Other topics:

* Terminology: • "Moneyness"
  • American-type claims: can be exercised <u>at or before</u> maturity
  • Greeks: "derivatives in the mathematical sense"
    delta $= \frac{\partial V}{\partial S}$
    gamma $= \left(\frac{\partial}{\partial S}\right)^2 V$
    vega (!) = derivative wrt σ etc.
  • Put option: right to <u>sell</u>
* Put/call parity:

$$C - P = S - AK$$

C: call price; P: put price; S: spot of stock; A: spot of zero-coupon bond paying 1 at option maturity

holds <u>regardless</u> of dynamics.

But: • same maturity everywhere, (else fails)
  • <u>invalid</u> for American-type claims

* Implied volatility: Take all prices, enter formula, solve wrt σ.
  Used to • correct <u>other</u> derivatives for mispricing
  • analyse model shortcomings
    → "volatility smile" ....