University of Oslo / Department of Economics / NCF April 3, 2012

ECON5160: Problems for March 27 and later

For March 27, make sure to do problem 1 and problems 3 (a) through (d). You should also be able to do the rest too, but since those problems are intended as background for the mathematics of finance, they can be postponed.

1 On Brownian motion

Let $\mathbf{B}(t) = (B_1(t), B_2(t), \ldots, B_n(t))$, where the B_i are iid. standard Brownian motions – this $\mathbf{B}$ is what we call standard n-dimensional Brownian motion.

(a) Let $\boldsymbol{\sigma}$ be a given n-vector. When is the dot product $\boldsymbol{\sigma} \cdot \mathbf{B}(t)$ a standard Brownian motion?

(b) Let $\boldsymbol{\sigma}(t)$ be an n-vector-valued *deterministic* function of t. When is $\int_0^t \boldsymbol{\sigma}(s) \cdot \mathrm{d}\mathbf{B}(s)$ a standard Brownian motion?

(c) If you got the same condition as I did, you should be able to generalize the previous point, to the case where $\boldsymbol{\sigma}$ is stochastic but independent of $\mathbf{B}$. (Hint: Double expectation.)

Comment: The common belief that sums of Gaussians are necessarily Gaussian, is too sloppy. Sums of *jointly* Gaussians are, sums of *marginally* Gaussians are not necessarily.

2 Self-financing portfolio composition

This problem will enable you to generate self-financing portfolios in frictionless markets. It requires no stochastic calculus except the fact that the ordinary product rule holds when one of the factors is «locally riskless», i.e. a $\mathrm{d}t$-integral.

Assume that we have a frictionless market consisting of $n+1$ investment opportunities whose prices at time t are denoted $Z_i(t)$. Denote your holdings of investment opportunity no. i at time t by $\nu_i(t)$ units. Then the total market value of your portfolio at time t, is

$$Y(t) = \nu_0(t)Z_0(t) + \nu_1(t)Z_1(t) + \cdots + \nu_n(t)Z_n(t).$$

Definition: An investment strategy is called *self-financing* if Y satisfies

$$\mathrm{d}Y(t) = \nu_0(t^-)\,\mathrm{d}Z_0(t) + \nu_1(t^-)\,\mathrm{d}Z_1(t) + \cdots + \nu_n(t^-)\,\mathrm{d}Z_n(t)$$

(this has the interpretation that wealth changes are due to price changes only, not to adding or deducting from the portfolio). Also, an investment strategy $Y(t) - C(t)$ is said to *finance consumption* $C(t)$ if Y is self-financing. For the rest of this problem, assume Y self-financing, and that the price dynamics may be written as

$$\mathrm{d}Z_i(t) = Z_i(t)\,\mathrm{d}X_i(t)$$

where Z_0 is «locally riskless» in the sense that $\mathrm{d}X_0(t) = r(t)\,\mathrm{d}t$ (consider r the riskless interest rate).

(a) Let $v_i(t)$ denote the market value of your position in opportunity no. i at time t. Show:

$$\mathrm{d}Y(t) = v_0(t)\,\mathrm{d}X_0(t) + v_1(t)\,\mathrm{d}X_1(t) + \cdots + v_n(t)\,\mathrm{d}X_n(t)$$

and that if $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ (no v_0 nor X_0!) and $\mathbf{1}$ is an n-vector of ones, then

$$\mathrm{d}Y(t) = Y(t)\,\mathrm{d}X_0(t) + \mathbf{v}(t)\cdot(\,\mathrm{d}\mathbf{X}(t) - \mathbf{1}\,\mathrm{d}X_0(t))$$

(Hint: $v_0 + \mathbf{v}\cdot\mathbf{1} = Y$.)

(b) Take Z_0 as numéraire, and put

$$\begin{aligned}\hat{Y}(t) &= Y(t)/Z_0(t)\\ \hat{\mathbf{v}}(t) &= \mathbf{v}(t)/Z_0(t)\\ \hat{\mathbf{X}}(t) &= \mathbf{X}(t) - \mathbf{1}X_0(t).\end{aligned}$$

Show that

$$\mathrm{d}\hat{Y}(t) = \hat{\mathbf{v}}(t)\cdot\,\mathrm{d}\hat{\mathbf{X}}(t). \tag{*}$$

3 Simple examples on Itô's formula – and why financial mathematics needs the second-order term

The headline says it all? OK, I threw in a not-so-simple example for good measure. Throughout this problem, assume X continuous.

First:

(a) In the formula $\mathrm{d}(XY) = Y\,\mathrm{d}X + X\,\mathrm{d}Y + (\,\mathrm{d}X)(\,\mathrm{d}Y)$, why is there no «$\frac{1}{2}$» in front of the latter term?

(b) Show that $(B(t))^2 - t$ is a martingale (hint: differentiate).

(c) Let F be a given function of one variable. Find a process $Z(t)$ which can be written as $\mathrm{d}Z = Y\,\mathrm{d}t$ and so that $F(B(t)) - Z(t)$ is a martingale. (Disregard the $\mathsf{E}|X| < \infty$ condition.)

(d) (Last part a bit tricky:) Itô's formula involves the second derivative, and needs to be modified for functions which are not twice continuously differentiable. Consider $A(t) = |B(t)|$.

   i. A process X is called a *supermartingale*[1] if for all $h > 0$, all t, we have $X(t) \geq \mathsf{E}[X(t+h)|X(\to t)]$, and a *submartingale* if the reverse inequality holds.
   Is A a supermartingale, a submartingale or neither? (Hint: Jensen's inequality.)

   ii. Show that for some U, we have $\mathrm{d}A = U(t)\,\mathrm{d}B$ on all intervals where A does not hit 0.

   iii. Explain why $Z(t) = A(t) - \int_0^t U(s)\,\mathrm{d}B(s)$ must be an increasing process, and why Z only increases when $A = 0$.

Now assume that X is a process which admits the *ordinary chain rule for differentiation* (i.e. no second-order term).

(e) Calculate $\int_0^T (X(t) - X(0))\,\mathrm{d}X(t)$ and show that it is nonnegative.

(f) Calculate $\int_0^T \max\{0, X(t) - X(0)\}\,\mathrm{d}X(t)$.
(Hint: Show that it is 0 if $X(T) \leq X(0)$, and therefore that you can take the lower integration limit to be $T_0 =$ the last time[2] for which $X \leq 0$.)

(g) Consider formula (*) above, where X and v are one-dimensional.

   i. Pick first $v(t) = X(t) - X(0)$. Why is X *useless* for the modeling purpose?

   ii. What about the case $v(t) = \max\{0, X(t) - X(0)\}$?

A comment: Not only does all differentiable X admit the ordinary chain rule in the integrations we performed; it follows (by the definition of the so-called Young integral) that if X is more than *half a time* differentiable, then a version of the above calculations are valid[3] and X is about just as useless for modeling frictionless financial markets.

[1] A «supermartingale» is expected to *decline!* The «super» thing about that, is that it always is as good as it gets.

[2] it does not matter that T_0 is not a stopping time.

[3] Shameless plug: Framstad: *«Arbitrage with the ordinary chain rule»*, Finance Letters 2(6), 2004

4 More Itô's formula – exponential martingales and pricing kernels

These problems are intended to prepare for the April 18th lecture; the G thing could be considered a stochastic component of the discount factor.

Let Z_0, Z and G satisfy

$$\begin{aligned}
\mathrm{d}Z_0(t) &= r(t)Z_0(t)\,\mathrm{d}t \\
\mathrm{d}Z(t) &= Z(t^-)\big(\mu(t)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B(t) + \gamma(t^-)\,\mathrm{d}\tilde{N}\big) \\
&= Z(t^-)\big((\mu(t) - \lambda(t)\gamma(t))\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B(t) + \gamma(t^-)\,\mathrm{d}N\big) \\
\mathrm{d}G(t) &= G(t^-)\big(-\theta(t)\,\mathrm{d}B(t) + \eta(t^-)\,\mathrm{d}\tilde{N}\big) \\
&= G(t^-)\big(-\lambda(t)\eta(t)\,\mathrm{d}t - \theta(t)\,\mathrm{d}B(t) + \eta(t^-)\,\mathrm{d}N\big)
\end{aligned}$$

where the coefficients are possibly stochastic, and where the martingale $\mathrm{d}\tilde{N}(t) = \mathrm{d}N(t) - \lambda(t)\,\mathrm{d}t$, where N is a *time-inhomogeneous* Poisson process with intensity $\lambda(t)$.

(a) First, show that for $\hat{Z}(t) = Z(t)/Z_0(t)$, we have

$$\mathrm{d}\hat{Z}(t) = \hat{Z}(t^-)\big((\mu(t) - r(t) - \lambda(t)\gamma(t))\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B(t) + \gamma(t^-)\,\mathrm{d}N\big)$$

(i.e. the same equation as Z satisfies, but with $\mu - r$ in place of μ.)

(b) Show that (suppressing time-dependence in some of the notation):

$$\begin{aligned}
&\mathrm{d}(\hat{Z}(t)G(t)) = \hat{Z}(t^-)G(t^-) \\
&\cdot\Big[(\mu - r - (\gamma+\eta)\lambda - \sigma\theta)\,\mathrm{d}t + (\sigma - \theta)\,\mathrm{d}B + (\eta(t^-) + \gamma(t^-) + \eta(t^-)\gamma(t^-))\,\mathrm{d}N\Big]
\end{aligned}$$

(c) Use this to give a condition on θ and η for $\hat{Z}G$ to be a martingale. (Disregard the $\mathsf{E}|\hat{Z}G| < \infty$ condition.)

Comment: For each, G defines a formal probability measure Q on events A known at time T, by $\mathsf{Pr}_Q[A] = \mathsf{E}_0[1_A G(T)]$, where actually 1_A – a random variable equalling one if A occurs and 0 otherwise – is an *Arrow–Debreu security.* In a typical financial mathematics paradigme, there is some measure Q, such that the price at time 0 of such a security,, can be represented as «$\mathsf{Pr}_Q[A]$ discounted as in the deterministic case» (i.e. with the price today of a zero-coupon bond maturing at T). For this reason, G is often referred to as a *pricing kernel.*