23.B.2. - A correction (I made a typo at the blackboard).

The buyer's problem is to report $\widehat{\theta}_2$ optimally.
The probability for trade is

$$\Pr\left[\widehat{\theta}_2 > \theta_1\right] = \widehat{\theta}_2.$$

The expected θ_1, conditional on trade:

$$\frac{0 + \widehat{\theta}_2}{2}.$$

The expected transfer, conditional on trade:

$$\mathrm{E}\frac{\widehat{\theta}_2 + \theta_1}{2} = \frac{\widehat{\theta}_2}{2} + \frac{1}{2}\frac{\widehat{\theta}_2}{2} = \frac{3}{4}\widehat{\theta}_2.$$

So, buyer's problem is:

$$\max_{\widehat{\theta}_2} \widehat{\theta}_2 \left(\theta_2 - \frac{3}{4}\widehat{\theta}_2\right).$$

FOC:

$$\begin{aligned} \theta_2 - \frac{3}{2}\widehat{\theta}_2 &= 0 \Rightarrow \\ \widehat{\theta}_2 &= \frac{2}{3}\theta_2. \end{aligned}$$

23.B.4 - Double Auction.

(a)
Seller is agent 1, and takes as given that 2's bid is of the following form:

$$b_2 = \alpha_2 + \beta_2\theta_2.$$

Given b_1, the probability that there is trade is

$$\begin{aligned} & \Pr\left[b_1 < b_2 = \alpha_2 + \beta_2\theta_2\right] \\ = & \Pr\left[\frac{b_1 - \alpha_2}{\beta_2} < \theta_2\right] \\ = & 1 - \frac{b_1 - \alpha_2}{\beta_2}. \end{aligned}$$

Furthermore, given that there is trade, we have

$$\mathrm{E}\theta_2 = \frac{1}{2}\left(1 + \frac{b_1 - \alpha_2}{\beta_2}\right).$$

Therefore, conditional on there being trade, the expected price is

$$\begin{aligned}
\mathrm{E}\frac{b_1+b_2}{2} &= \mathrm{E}\frac{b_1+\alpha_2+\beta_2\theta_2}{2} \\
&= \frac{b_1+\alpha_2}{2}+\frac{\beta_2}{2}\left[\frac{1}{2}\left(1+\frac{b_1-\alpha_2}{\beta_2}\right)\right] \\
&= \frac{3}{4}b_1+\frac{1}{4}\alpha_2+\frac{\beta_2}{4}.
\end{aligned}$$

The expected value conditional on trade is, to the seller, thus:

$$\frac{3}{4}b_1+\frac{1}{4}\alpha_2+\frac{\beta_2}{4}-\theta_1.$$

By combining these things, we can see that the expected payoff to the seller is:

$$\left(1-\frac{b_1-\alpha_2}{\beta_2}\right)\left(\frac{3}{4}b_1+\frac{1}{4}\alpha_2+\frac{\beta_2}{4}-\theta_1\right).$$

FOC:

$$\begin{aligned}
-\frac{1}{\beta_2}\left(\frac{3}{4}b_1+\frac{1}{4}\alpha_2+\frac{\beta_2}{4}-\theta_1\right)+\frac{3}{4}\left(1-\frac{b_1-\alpha_2}{\beta_2}\right) &= 0 \Rightarrow \\
-\left(\frac{1}{4}\alpha_2+\frac{\beta_2}{4}-\theta_1\right)+\frac{3}{4}\left(\beta_2+\alpha_2\right) &= \frac{3}{2}b_1 \Rightarrow
\end{aligned}$$

$$b_1=\frac{2}{3}\theta_1+\frac{1}{3}\alpha_2+\frac{1}{3}\beta_2.$$

Even though $\beta_1=2/3$, the seller is likely to bid $b_1>\theta_1$ because of the constant parts. For example, if the buyer were to bid "truthfully, in that $\alpha_2=0$ and $\beta_2=1$, then $b_1=\frac{2}{3}\theta_1+\frac{1}{3}>\theta_1$ whenever $\theta_1<1$.

From the first part, on the blackboard, we already derived (correctly!):

$$b_2=\frac{2}{3}\theta_2 \text{ and } \alpha_2=\frac{\alpha_1}{3}.$$

Combined,

$$\begin{aligned}
\alpha_1 &= \frac{1}{3}\alpha_2+\frac{1}{3}\beta_2 \\
&= \frac{1}{3}\frac{\alpha_1}{3}+\frac{1}{3}\frac{2}{3} \\
&= \frac{1}{4}, \text{ so} \\
\alpha_2 &= \frac{1}{12}.
\end{aligned}$$

Therefore

$$\begin{aligned}
b_1 &= \frac{2}{3}\theta_1+\frac{1}{4}, \\
b_2 &= \frac{2}{3}\theta_2+\frac{1}{12}.
\end{aligned}$$

There will be trade only when

$$\begin{aligned} b_2 &> b_1 \Rightarrow \\ \frac{2}{3}\theta_2 + \frac{1}{12} &> \frac{2}{3}\theta_1 + \frac{1}{4} \Rightarrow \\ \theta_2 &> \theta_1 + \frac{1}{4}. \end{aligned}$$

So, there is too little trade! It would be efficient to trade whenever $\theta_2 > \theta_1$.

This implements the following social choice function: f is such that trade takes place if an only if $\theta_2 > \theta_1 + \frac{1}{4}$ and, then, the transfer from the buyer to the seller is:

$$t = \frac{b_1 + b_2}{2} = \frac{1}{2}\left(\frac{2}{3}\theta_1 + \frac{1}{4} + \frac{2}{3}\theta_2 + \frac{1}{12}\right) = \frac{\theta_1 + \theta_2}{3} + \frac{1}{6}.$$

This is NOT ex post efficient, since trade does NOT take place, even if that would have been ex post optimal, if $\theta_2 \in (\theta_1, \theta_1 + 1/4)$.

(b) The social choice function just given can be truthfully implemented if one player's best respose is to report truthfully conditional on the other reporting truthfully. We should check that this is the case for both parts.

For the seller, reporting $\widehat{\theta}_1$, assuming buyer reports type truthfully:

Probability for trade:

$$\Pr\left(\theta_2 > \widehat{\theta}_1 + \frac{1}{4}\right) = 1 - \widehat{\theta}_1 - \frac{1}{4} = \frac{3}{4} - \widehat{\theta}_1.$$

Expected θ_2, conditional on trade:

$$\frac{1}{2}\left(1 + \widehat{\theta}_1 + \frac{1}{4}\right) = \frac{1}{2}\left(\widehat{\theta}_1 + \frac{5}{4}\right).$$

Expected transfer, conditional on trade:

$$\mathrm{E}\frac{\widehat{\theta}_1 + \theta_2}{3} + \frac{1}{6} = \frac{\widehat{\theta}_1}{3} + \frac{1}{6} + \frac{1}{3}\frac{1}{2}\left(\widehat{\theta}_1 + \frac{5}{4}\right) = \frac{\widehat{\theta}_1}{2} + \frac{9}{24}.$$

Seller's problem:

$$\max_{\widehat{\theta}_1} \left(\frac{3}{4} - \widehat{\theta}_1\right)\left(\frac{\widehat{\theta}_1}{2} + \frac{9}{24} - \theta_1\right).$$

FOC:

$$\begin{aligned} -\left(\frac{\widehat{\theta}_1}{2} + \frac{9}{24} - \theta_1\right) + \frac{1}{2}\left(\frac{3}{4} - \widehat{\theta}_1\right) &= 0 \Rightarrow \\ \theta_1 - \frac{9}{24} + \frac{3}{8} &= \widehat{\theta}_1 \Rightarrow \theta_1 = \widehat{\theta}_1. \end{aligned}$$

Similarly, we can prove that for the buyer, it is optimal with $\widehat{\theta}_2 = \theta_2$.