Topics in Microeconomics ECON 5210 (Part II - Contract Theory)

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Moral Hazard

Introduction

The principal P hires an agent A to perform a task. A chooses the effort level, but P can only observe the performance, which is a noisy reflection of effort.

Theoretical development

- Holmström 79, 83, Grossman and Hart 83

Application

- Theory of firm (Grossman and Hart 82)
- Sharecropping contract (Stiglitz 74)

Two performance outcome

Framework

Performance $q \in \{0, 1\}$

Action $a \geq 0$

$\Pr(q = 1 \mid a) = p(a)$, increasing, concave in a;

Assume $p(0) = 0, p(\infty) = 1$ and $p'(0) > 1$

P offers wage w to agent A

P gets $V(q - w)$, V increasing and concave

A gets $u(w) - a$, u increasing and concave

Suppose action is observable and verifiable so that P is perfectly aware of the individual rationality constraint of A, conditioning on a

$\max_{a,w_i} p(a) V(1 - w_1) + (1 - p(a)) V(-w_0)$

subject to $p(a) u(w_1) + (1 - p(a)) u(w_0) - a \geq 0$

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$\max_{a,w_i} p(a)\, V(1 - w_1) + (1 - p(a))\, V(-w_0)$

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Individual rationality condition binding

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Individual rationality condition binding

First order condition with respect to w_i: Optimal coinsurance (Borch rule)

$\frac{V'(1-w_1)}{u'(w_1)} = \lambda = \frac{V'(-w_0)}{u'(w_0)}$

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$\frac{V'(1-w_1)}{u'(w_1)} = \lambda = \frac{V'(-w_0)}{u'(w_0)}$

First order condition with respect to effort

$p'(a)\left[V(1-w_1) - V(-w_0)\right] + \lambda p'(a)\left[u(w_1) - u(w_0)\right] - \lambda = 0$

- Risk neutral principal $V(x) = x$

$$w_1^* = w_0^* = w^*; p'(a^*) = \frac{1}{u'(w^*)}; u(w^*) = a^*$$

P bearing all the risk

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$$w_1^* = w_0^* = w^*; p'(a^*) = \frac{1}{u'(w^*)}; u(w^*) = a^*$$

P bearing all the risk

- Risk neutral agent $u(x) = x$

$$w_1^* - w_0^* = 1; p'(a^*) = 1$$

P selling the project to A at an upfront price $-w_0^*$

$$\max_{a,w_i} p(a) V(1 - w_1) + (1 - p(a)) V(-w_0)$$

subject to

$$IR : p(a) u(w_1) + (1 - p(a)) u(w_0) - a \geq 0$$

$$IC : a \in \arg\max_{\widehat{a}} p(\widehat{a}) u(w_1) + (1 - p(\widehat{a})) u(w_0) - \widehat{a}$$

$\max_{a,w_i} p(a)\, V(1 - w_1) + (1 - p(a))\, V(-w_0)$

subject to

$IR : p(a)\, u(w_1) + (1 - p(a))\, u(w_0) - a \geq 0$

$IC : a \in \arg\max_{\widehat{a}} p(\widehat{a})\, u(w_1) + (1 - p(\widehat{a}))\, u(w_0) - \widehat{a}$

$IC : p'(a)\, [u(w_1) - u(w_0)] = 1$

Second best: Bilateral risk neutrality

Effect of agent's risk neutrality $(u(x) = x)$

$IC \Rightarrow p'(a)[w_1 - w_0] = 1$

First best $a = a^*$ can still be implemented if we could have $[w_1 - w_0] = 1$

Easy to check that IR will be satisfied as $p(a^*) + w_0 - a^* \geq 0$, so P can effectively charge $w_0^* = p(a^*) - a^*$

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What if A has resource constraint? For example, what if $w_0 \geq 0$? Will the above contract be still optimal?

Second best: Bilateral risk neutrality

Consider P's problem with $w_0 \geq 0$ (ignore the IR constraint for the time being and look at the optimal effort choice)

$\max_{a,w_i} p(a)(1 - w_1) + (1 - p(a))(-w_0)$

subject to

$IC : p'(a)[w_1 - w_0] = 1$

Optimal to charge $w_0 = 0$; Solve for w_1 and a

$p'(a) = 1 - \frac{p(a)p''(a)}{[p'(a)]^2}$

Even with risk neutrality, there is distortion in terms of effort provision, solely due to unobservable action

Second best: Bilateral risk aversion

$\max_{a,w_i} p(a) V(1-w_1) + (1-p(a)) V(-w_0)$

subject to

$IR : p(a) u(w_1) + (1-p(a)) u(w_0) - a \geq 0$

$IC : p'(a) [u(w_1) - u(w_0)] = 1$

First order conditions

$\frac{V'(1-w_1)}{u'(w_1)} = \lambda + \mu \frac{p'(a)}{p(a)}$

$\frac{V'(-w_0)}{u'(w_0)} = \lambda - \mu \frac{p'(a)}{[1-p(a)]}$

Coinsurance scheme distorted

A gets more (less) compared to Borch rule in case of high (low) performance. This distortion is to induce effort from *A* by rewarding for 'success' whose frequency rises with effort.

Linear contract

Assume risk neutral P and agents with CARA preference

$q = a + \varepsilon; \varepsilon \sim N\left(0, \sigma^2\right)$

$u\left(a, w\right) = -e^{-\eta[w - \psi(a)]}$

$w = t + sq$

P's problem

$\max_{a,t,s} E\left(q - w\right)$

subject to

$IR : E\left(-e^{-\eta[w-\psi(a)]}\right) \geq u\left(\overline{w}\right)$

$IC : a = \arg\max_{\tilde{a}} E\left(-e^{-\eta[w-\psi(\tilde{a})]}\right)$

Agent's choice of action

$$a = \arg\max_{\tilde{a}} E\left(-e^{-\eta\left[t+s\tilde{a}+s\varepsilon-\frac{c\tilde{a}^2}{2}\right]}\right)$$

$$= \arg\max_{\tilde{a}} \left[-e^{-\eta\left[t+s\tilde{a}+s\varepsilon-\frac{c\tilde{a}^2}{2}\right]}\right] E\left(-e^{\eta s\varepsilon}\right)$$

$$= \arg\max_{\tilde{a}} - e^{-\eta\left[t+s\tilde{a}-\frac{c\tilde{a}^2}{2}-\frac{\eta}{2}s^2\sigma^2\right]}$$

$= \arg\max_{\tilde{a}} - e^{-\eta\widehat{w}(\tilde{a})}$ where $\widehat{w}(\tilde{a})$ is the *certainty equivalent compensation*

$$= \arg\max_{\tilde{a}} \widehat{w}(\tilde{a}) = \frac{s}{c} = \widehat{a}, \text{ say}$$

P's problem

$$max_{t,s} E\left(\frac{s}{c} + \varepsilon - t - \frac{s^2}{c} - s\varepsilon\right)$$

subject to

$$IR : \widehat{w}\left(\frac{s}{c}\right) = \overline{w}$$

P's problem

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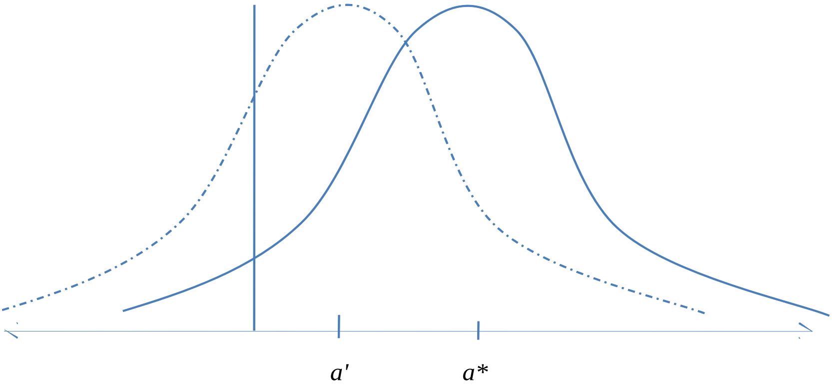
Optimal $\widehat{s} = \frac{1}{1+\eta c \sigma^2}$; decreasing in c, η and σ^2

Sub-optimality of linear contract

$q = a + \varepsilon$



Multitasking: Holmström and Milgrom 91

A undertakes two tasks a_1 and a_2

$q_i = a_i + \varepsilon_i$; $(\varepsilon_1, \varepsilon_2) \sim$ multivariate normal with zero mean and covariance matrix $\sum = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$

$u(w, a) = -e^{-\eta[w - \psi(a_1, a_2)]}$

where $\psi(a_1, a_2) = \frac{1}{2}\left(c_1 a_1^2 + c_2 a_2^2\right) + \delta a_1 a_2; \delta \in \left[0, \sqrt{c_1 c_2}\right]$

Linear contracts: $w = t + s_1 q_1 + s_2 q_2$

Solving agent's choice of action

Determine certainty equivalent

$$\widehat{w}\left(a_1, a_2\right) = t + s_1 a_1 + s_2 a_2 - \frac{\eta}{2}\left(s_1^2\sigma_1^2 + s_2^2\sigma_2^2\right) - \frac{1}{2}\left(c_1 a_1^2 + c_2 a_2^2\right) - \delta a_1 a_2$$

Solving agent's choice of action

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Solving for a_1 and a_2

$$s_i = c_i a_i + \delta a_j$$

Solving agent's choice of action

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Solving for a_1 and a_2

$s_i = c_i a_i + \delta a_j$

$a_i = \frac{s_i c_j - \delta s_j}{c_i c_j - \delta^2}$

The Principal's problem

$$\max a_1 (1 - s_1) + a_2 (1 - s_2) - t$$

subject to

$$a_1 = \frac{s_1 c_2 - \delta s_2}{c_1 c_2 - \delta^2}, a_2 = \frac{s_2 c_1 - \delta s_1}{c_2 c_1 - \delta^2}$$

The Principal's problem

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$$IR : \widehat{w}(a_1, a_2) =$$

$$t + s_1 a_1 + s_2 a_2 - \frac{\eta}{2}\left(s_1^2 \sigma_1^2 + s_2^2 \sigma_2^2\right) - \frac{1}{2}\left(c_1 a_1^2 + c_2 a_2^2\right) - \delta a_1 a_2 \geq \overline{w}$$

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$$a_1 = \frac{s_1 c_2 - \delta s_2}{c_1 c_2 - \delta^2}, a_2 = \frac{s_2 c_1 - \delta s_1}{c_2 c_1 - \delta^2}$$

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$$t + s_1 a_1 + s_2 a_2 - \frac{\eta}{2}\left(s_1^2 \sigma_1^2 + s_2^2 \sigma_2^2\right) - \frac{1}{2}\left(c_1 a_1^2 + c_2 a_2^2\right) - \delta a_1 a_2 \geq \overline{w}$$

Easy to see that IR is binding

Replace t in P's objective function, and check the first order condition

$$s_i^* = \frac{1 + \left(c_j - \delta\right)\eta\sigma_j^2}{1 + \eta c_i \sigma_i^2 + \eta c_j \sigma_j^2 + \eta^2 \sigma_i^2 \sigma_j^2 \left(c_i c_j - \delta^2\right)}, i = 1, 2$$

Discussion

- If $\delta = 0$,or two tasks are technologically independent
  - We will have $a_i^* = \frac{s_i^*}{c_i}$ and $s_i^* = \frac{1}{1+\eta c_i \sigma_i^2}$

Discussion

- If $\delta = 0$,or two tasks are technologically independent
  - We will have $a_i^* = \frac{s_i^*}{c_i}$ and $s_i^* = \frac{1}{1+\eta c_i \sigma_i^2}$
- Endogenous complementarity between s's
  - $\frac{\partial}{\partial\left(\sigma_1^2\right)} s_2 < 0$

Discussion

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  - We will have $a_i^* = \frac{s_i^*}{c_i}$ and $s_i^* = \frac{1}{1+\eta c_i \sigma_i^2}$
- Endogenous complementarity between s's
  - $\frac{\partial}{\partial\left(\sigma_1^2\right)} s_2 < 0$
  - What if P has little control over s_2?

Framework

Risk neutral shareholders (P) hiring a risk neutral CEO (A) to run the company

Two possible realization $X \in \{0, R\}; p(X = R) = \theta e$

Ability $\theta = \theta_L$ and θ_H with respective probabilities β and $1 - \beta$

Effort is costly: $\psi(e) = \frac{c}{2}e^2$

Contract: (t_i, r_i) where t_i is an upfront payment and r_i is the repayment amount after revenue is realized (from A to P)

A's choice of effort

(t_i, r_i) given

$\max_e \theta_i e (R - r_i) - t_i - \frac{c}{2} e^2, i = L, H$

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(t_i, r_i) given

$\max_e \theta_i e (R - r_i) - t_i - \frac{c}{2}e^2, i = L, H$

Optimal effort (solving the first order condition)

$e_i = \frac{\theta_i(R-r_i)}{c}$

Maximum payoff: $\frac{1}{2c}\left[\theta_i (R - r_i)\right]^2 - t_i$

P's optimization problem

$$\max_{(t_i,r_i)} \left\{\beta \left[t_H + \theta_H^2 \left(R - r_H\right) \frac{r_H}{c}\right] + (1-\beta) \left[t_L + \theta_L^2 \left(R - r_L\right) \frac{r_L}{c}\right]\right\}$$

$$IC : \frac{1}{2c} \left[\theta_i \left(R - r_i\right)\right]^2 - t_i \geq \frac{1}{2c} \left[\theta_i \left(R - r_j\right)\right]^2 - t_j ; j \neq i, i = L, H$$

$$IR : \frac{1}{2c} \left[\theta_i \left(R - r_i\right)\right]^2 - t_i \geq 0, i = L, H$$

Optimal contract with moral hazard only

Suppose P knows A's ability

$\max_{(t_i, r_i)} \left[t_i + \theta_i^2 \left(R - r_i \right) \frac{r_i}{c} \right]$

$IR : \frac{1}{2c} \left[\theta_i \left(R - r_i \right) \right]^2 - t_i \geq 0, i = L, H$

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IR is binding; Replace t_i in the objective function

$\max_{r_i} \frac{1}{2c} \theta_i^2 \left(R^2 - r_i^2 \right)$

Optimal contract with moral hazard only

Suppose P knows A's ability

$\max_{(t_i,r_i)} \left[t_i + \theta_i^2 (R - r_i) \frac{r_i}{c}\right]$

$IR : \frac{1}{2c} [\theta_i (R - r_i)]^2 - t_i \geq 0, i = L, H$

IR is binding; Replace t_i in the objective function

$\max_{r_i} \frac{1}{2c}\theta_i^2 \left(R^2 - r_i^2\right)$

Solution: $r_H = r_L = 0; t_i = \frac{\theta_i^2 R^2}{2c}$, P sells the firm to A

Optimal contract with adverse selection only

Suppose A provides a fixed effort $\overline{e}$, but P cannot observe θ

$\max_{(t_i, r_i)} \left\{ \beta \left[t_H + \theta_H \overline{e} r_H \right] + (1-\beta) \left[t_L + \theta_L \overline{e} r_L \right] \right\}$

$IC : \theta_i \overline{e} \left(R - r_i \right) - t_i \geq \theta_i \overline{e} \left(R - r_j \right) - t_j$

$IR : \theta_i \overline{e} \left(R - r_i \right) - t_i - \frac{c}{2} \overline{e}^2 \geq 0$

Optimal contract with adverse selection only

Suppose A provides a fixed effort $\overline{e}$, but P cannot observe θ

$\max_{(t_i,r_i)} \{\beta [t_H + \theta_H \overline{e} r_H] + (1-\beta) [t_L + \theta_L \overline{e} r_L]\}$

$IC : \theta_i \overline{e} (R - r_i) - t_i \geq \theta_i \overline{e} (R - r_j) - t_j$

$IR : \theta_i \overline{e} (R - r_i) - t_i - \frac{c}{2} \overline{e}^2 \geq 0$

Solution: $r_L = r_H = R, t_i = -\frac{c}{2} \overline{e}^2$; Complete extraction of the informational rent

Optimal contract with both moral hazard and adverse selection

Note that *IRL* and *ICH* will be binding; High type's informational rent is bounded below by its utility if it had accepted the offer made for the low type

Optimal contract with both moral hazard and adverse selection

Note that IRL and ICH will be binding; High type's informational rent is bounded below by its utility if it had accepted the offer made for the low type

$$\max_{(t_i, r_i)} \left\{ \beta \left[t_H + \theta_H^2 (R - r_H) \frac{r_H}{c} \right] + (1 - \beta) \left[t_L + \theta_L^2 (R - r_L) \frac{r_L}{c} \right] \right\}$$

$$IC : \frac{1}{2c} \left[\theta_H (R - r_H) \right]^2 - t_H = \frac{1}{2c} \left[\theta_H (R - r_L) \right]^2 - t_L$$

$$IR : \frac{1}{2c} \left[\theta_L (R - r_L) \right]^2 - t_L = 0$$

Optimal contract with both moral hazard and adverse selection

Note that *IRL* and *ICH* will be binding; High type's informational rent is bounded below by its utility if it had accepted the offer made for the low type

$$\max_{(t_i, r_i)} \left\{\beta\left[t_H + \theta_H^2 (R - r_H) \frac{r_H}{c}\right] + (1-\beta)\left[t_L + \theta_L^2 (R - r_L) \frac{r_L}{c}\right]\right\}$$

$$IC : \frac{1}{2c}\left[\theta_H (R - r_H)\right]^2 - t_H = \frac{1}{2c}\left[\theta_H (R - r_L)\right]^2 - t_L$$

$$IR : \frac{1}{2c}\left[\theta_L (R - r_L)\right]^2 - t_L = 0$$

Solution: $r_L = 0$;

Optimal contract with both moral hazard and adverse selection

Note that *IRL* and *ICH* will be binding; High type's informational rent is bounded below by its utility if it had accepted the offer made for the low type

$$\max_{(t_i, r_i)} \left\{ \beta \left[t_H + \theta_H^2 (R - r_H) \tfrac{r_H}{c} \right] + (1-\beta) \left[t_L + \theta_L^2 (R - r_L) \tfrac{r_L}{c} \right] \right\}$$

$$IC : \tfrac{1}{2c} \left[\theta_H (R - r_H) \right]^2 - t_H = \tfrac{1}{2c} \left[\theta_H (R - r_L) \right]^2 - t_L$$

$$IR : \tfrac{1}{2c} \left[\theta_L (R - r_L) \right]^2 - t_L = 0$$

Solution: $r_L = 0$; $r_H = \frac{\beta\left(\theta_H^2 - \theta_L^2\right)R}{\beta\left(\theta_H^2 - \theta_L^2\right) + (1-\beta)\theta_L^2}$; No effort distortion for the high ability A, but downward effort distortion for the low ability A

References and exercises

- Moral hazard
  - Two performance case (BD Ch. 4.1)
  - Linear contracts (BD Ch. 4.2)
  - Multiple tasks (BD Ch. 6.2)
- Combining moral hazard and adverse selection (BD Ch. 6.3)
- Problems BD 4.4, BD 6.2